

On the geometry of the supermultiplet in M-theory

Hisham Sati *

*Department of Mathematics
Yale University
New Haven, CT 06511*

*Department of Mathematics
University of Maryland
College Park, MD 20742*

Abstract

The massless supermultiplet of eleven-dimensional supergravity can be generated from the decomposition of certain representation of the exceptional Lie group F_4 into those of its maximal compact subgroup $\text{Spin}(9)$. In an earlier paper, a dynamical Kaluza-Klein origin of this observation is proposed with internal space the Cayley plane, $\mathbb{O}P^2$ and topological aspects are explored. In this paper we consider the geometric aspects and propose a characterization of the origin of the massless fields and their supersymmetry in M-theory. The effect of the construction on the partition function and the compatibility with other physical theories is discussed.

*E-mail: hisham.sati@yale.edu

Contents

1	Introduction	1
2	The Fields in M-theory	3
2.1	The Euler Triplet	3
2.2	Spin(9)-structures and the M-theory fields	5
2.2.1	Spin(9) bundles	5
2.2.2	Spin(9)-structures	6
2.2.3	Consequences for the M-theory fields	9
2.3	Supersymmetry	10
2.4	Relating Y^{11} and M^{27}	12
2.4.1	geometric consequences	12
2.5	Structures on M^{27}	14
3	Terms in the Lifted Action	17
3.1	The Cayley 8-form	17
3.2	Torsion classes and effect on the M-theory partition function	19
3.2.1	Classes from BF_4	20
3.2.2	Classes from $\mathbb{O}P^2$	21
3.3	Further terms and compatibility with other theories	22
3.3.1	Kinetic terms	22
3.3.2	Compatibility with ten-dimensional superstring theories	23
3.3.3	Compatibility with the bosonic string	25
4	Appendix: Some Properties of $\mathbb{O}P^2$	26

1 Introduction

We propose an origin of the massless multiplet in M-theory as Cayley plane bundles $\mathbb{O}P^2$ over eleven-dimensional spacetime. This is a continuation of the paper [55], where topological and number-theoretic aspects were explored. In this paper we focus on the geometric aspects and discuss some implications on physical constructs, such as the partition function and supersymmetry.

The eleven-dimensional massless supermultiplet (g, C_3, Ψ) , composed of the metric g , the C -field C_3 , and the Rarita-Schwinger field Ψ , is related to F_4 , the exceptional Lie group of rank 4. Ramond [50] [51] [52] gave evidence for F_4 coming from the following two related observations:

1. F_4 appears explicitly [52] in the light-cone formulation of supergravity in eleven dimensions [17]. The generators $T^{\mu\nu}$ of the little group $SO(9)$ of the Poincaré group $ISO(1, 10)$ in eleven dimensions and the spinor generators T^a combine to form the 52 operators that generate the exceptional Lie algebra \mathfrak{f}_4 such that the constants $f^{\mu\nu ab}$ in the commutation relation

$$[T^{\mu\nu}, T^a] = i f^{\mu\nu ab} T^b \quad (1.1)$$

are the structure constants of \mathfrak{f}_4 . The 36 generators $T^{\mu\nu}$ are in the adjoint of $SO(9)$ and the 16 T^a generate its spinor representation. This can be viewed as the analog of the construction of E_8 out of the generators of $SO(16)$ and of $E_8/SO(16)$ in [27].

2. The identity representation of F_4 , i.e. the one corresponding to Dynkin index $[0, 0, 0, 0]$, generates the three representations of $\text{Spin}(9)$ [50] $\text{Id}(F_4) \longrightarrow (44, 128, 84)$, the numbers on the right hand side correctly matching the number of degrees of freedoms of the massless bosonic content of eleven-dimensional supergravity with the individual summands corresponding, respectively, to the graviton, the gravitino, and the C -field (see the beginning of section 2).

The main Idea of this paper, presented in section 2.1 is interpreting Ramond's triplets as arising from $\mathbb{O}P^2$ bundles with structure group F_4 over our eleven-dimensional manifold Y^{11} , on which M-theory is 'defined'. We first discuss in section 2.2 the geometric properties of $\mathbb{O}P^2$, including $\text{Spin}(9)$ -structures and characteristic classes. This leads to one of the main results, theorem 2.9, that the massless fields of M-theory are encoded in the spinor bundle of $\mathbb{O}P^2$. We then relate $\text{Spin}(9)$ -structures on the 9-dimensional vector space V^9 to the geometry of the eight-sphere S^8 which in turn, by [23], is related to Killing spinors on the cone over S^8 . This shows that the unification of the fields, as well as their supersymmetry, in M-theory can be seen from the eight-sphere over the Cayley plane (cf. proposition 2.11). We then show that fields can be given yet another interpretation via the index of the twisted (Kostant) Dirac operator of [43]. The identity representation of F_4 encoding the supergravity multiplet is the space of twisted harmonic spinors on $\mathbb{O}P^2$, which is proposition 2.12.

After studying structures on $\mathbb{O}P^2$, we use that space itself as the fiber over eleven-dimensional spacetime. In section 2.4.1 we explore the consequences of this idea by relating the geometry and the characteristic classes of the base to that of the total space, using the knowledge of that of the fiber studied in section 2. If the base Y^{11} has positive Ricci curvature then so does the total space M^{27} . This and related matters are discussed in section 2.4.1. In section 2.5 we relate structures, such as Fivebrane structures [56] [57], as well as genera on the base space to genera on the total space. This includes elliptic genera, Witten genera, Ochanine genera and is the content of proposition 2.15. We use this to relate the genera to an elliptic refinement of the mod index of the Dirac operator appearing in the study of the M-theory partition function [19] [41].

In section 3 we consider possible terms in the lifted action up in twenty-seven dimensions. In particular, in section 3.1 we consider the Cayley 8-form, which is a generalization to manifolds of $\text{Spin}(9)$ holonomy of the Cayley 4-form or the Kähler 2-form on manifolds with quaternionic and complex structures, and identify that 8-form as a representative in the cohomology of the Cayley plane and as a possible term in the lifted action. Then in section 3.2 we consider torsion classes and their effect on the M-theory partition function. We show in propositions 3.1 and 3.2 that \mathbb{Z}_2 and \mathbb{Z}_3 classes from the classifying space BF_4 are compatible with the M-theory partition function.

In section 3.3 we consider further possible terms in twenty-seven dimensions. In particular, in section 3.3.1 we consider possible terms, lifting the degree eight class introduced in [18], and which generalize the $G_4 \wedge *G_4$ term in the eleven-dimensional action. A natural question then arises whether the construction in this paper is compatible with type II superstring theory in ten dimensions and bosonic string theory in twenty-six dimensions. We study the former in section 3.3.2, where we show that the dimensional reduction of the F_4 bundle on the circle in Y^{11} leads to an LF_4 bundle over X^{10} and, under some natural assumptions, compatibility with type II string theory. We discuss the latter, i.e. the compatibility with bosonic string theory, in section 3.3.3. Finally we collect in the appendix some basic useful properties of the Cayley plane.

We use the Lorentz signature in studying the spectrum in section 2, and then resort to the Euclidean signature when discussing the geometric aspects in the rest of the paper.

2 The Fields in M-theory

The low energy limit of M-theory (cf. [60] [59] [21]) is eleven-dimensional supergravity [17], whose field content on an eleven-dimensional spin manifold Y^{11} with Spin bundle SY^{11} is

- **Two bosonic fields:** The metric g and the three-form C_3 . It is often convenient to work with Cartan's moving frame formalism so that the metric is replaced by the 11-bein e_M^A such that $e_M^A e_N^B = g_{MN} \eta^{AB}$, where η is the flat metric on the tangent space.
- **One fermionic field:** The Rarita-Schwinger vector-spinor Ψ_1 , which is classically a section of $SY^{11} \otimes TY^{11}$, i.e. a spinor coupled to the tangent bundle.

The count of the on-shell degrees of freedom, i.e. components, of the fields is done by eliminating the redundant gauge degrees of freedom. This could be done for example by choosing the light cone gauge: decompose Minkowski space $\mathbb{R}^{1,10}$ into $\mathbb{R}^{1,1} \oplus \mathbb{R}^9$, with $\mathbb{R}^{1,1} = \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ where the vectors \mathbf{v}_i satisfy $|\mathbf{v}_1|^2 = |\mathbf{v}_2|^2 = 0$ and $\mathbf{v}_1 \cdot \mathbf{v}_2 \neq 0$.

The Poincaré group $\mathbb{R}^{1,10} \ltimes SO(1,10)$ corresponds to the algebra $\mathbb{R}^{1,10} \widetilde{\oplus} \mathfrak{so}(1,10)$ where the brackets $[\mathbb{R}^{1,10}, \mathfrak{so}(1,10)]$ are given by the vector representation of $\mathfrak{so}(1,10)$ on $\mathbb{R}^{1,10}$. Since the latter is abelian then the irreducible representations are one-dimensional, and hence given by the characters $(\mathbb{R}^{1,10})^*$. This is acted upon by $\mathfrak{so}(1,10)$, which decomposes the space of characters into orbits characterized by the mass $m^2 = |\mathbf{v}|^2$ for $\mathbf{v} \in (\mathbb{R}^{1,10})^*$. Let H be the stabilizer of a point. H is called the little group. An irreducible representation of the Poincaré algebra is the space of sections of a homogeneous vector bundle $E = SO(1,10) \times_H K$ over the orbit $SO(1,10)/H$, where K is a representation of H . The representations, by the Wigner classification, are as follows:

- *Massive fields:* For $|\mathbf{v}|^2 \neq 0$ the little group is $H = SO(10)$.
- *Massless fields:* For $|\mathbf{v}|^2 = 0$ the little group is $H = SO(9)$.

The states for eleven-dimensional supergravity are massless and hence form irreducible representations of the little group $SO(9)$. The count is as follows (with $D = 11$):

1. *The 11-bein e_M^A :* Traceless symmetric $(D-2) \times (D-2)$ matrix gives $\frac{1}{2}D(D-3) = 44$ [35].
2. *The C -field C_3 :* A 3-form in \mathbb{R}^9 gives $\binom{D-2}{3} = \frac{(D-2)!}{3!(D-2-3)!} = 84$.
3. *The Rarita-Schwinger field Ψ_1 :* $2^{\frac{1}{2}(D-1)-1}(D-3) = 128$, where the factor of -1 in the exponent comes from the fact that Ψ_1 is a Majorana, i.e. real, fermion.

2.1 The Euler Triplet

In this section we review Ramond's observation we mentioned in the introduction and state the main theme of this paper. We will basically 'geometrize' and 'topologize' the representation-theoretic observation, hence making room for dynamics from kinematics. Therefore, the appearance of the F_4 representation and the decomposition under the maximal compact subgroup $\text{Spin}(9)$ to give the degrees of freedom of the fields will be taken to originate from an $\mathbb{O}P^2$ bundle over Y^{11} .

There are anomalous embeddings of certain groups into an orthogonal group in which the vector representation of the bigger group is identified with the spinor of the smaller group. For example, for $SO(9)$ we

have [35]

$$\begin{aligned} SO(16) &\supset SO(9) \\ \text{vector} &= \text{spinor}, \end{aligned} \quad (2.1)$$

both of dimension 16. In fact this explains the emergence of supersymmetry for the supermultiplet of eleven-dimensional supergravity [35] [20] [50]. Furthermore, in [20] it was conjectured that $SO(16)$ is a local symmetry of 11-dimensional supergravity. This was proved in [48]. One of the goals in this paper will be to seek a geometric origin for the above observation (eqn. (2.1)) via $\mathbb{O}P^2$ bundles, as $\text{Spin}(16)$ will be the Spin group of the projective plane fiber. We hope this would also shed some light on the enlarged local symmetry in the theory since the symmetry groups coming from bundles on $\mathbb{O}P^2$ will act locally (at least on the space itself).

Since $\text{rank}(F_4) = \text{rank}(\text{Spin}(9))$ then $\mathbb{O}P^2$ is an equal rank symmetric space. A generalization to homogeneous spaces of the Weyl character formula, with maximal torus replaced by the equal rank maximal compact subgroup, is the Gross-Kostant-Ramond-Sternberg character formula [28]

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_c \text{sgn}(c) U_{c \bullet \lambda}, \quad (2.2)$$

which can be applied as follows [50] to the pair $(F_4, \text{Spin}(9))$. The left hand side involves the differences of tensor products of representations V_λ of F_4 with highest weight λ written in terms of its $\text{Spin}(9)$ subgroup, and S^\pm , the two semi-spinor representations of $\text{Spin}(16)$ written in terms of its embedded subgroup $\text{Spin}(9)$, i.e. the spin representation associated to the complement of $\mathfrak{spin}(9) = \text{Lie}(\text{Spin}(9))$ in $\mathfrak{f}_4 = \text{Lie}(F_4)$. The right hand side involves the sum over c , the elements of the Weyl group which map the Weyl chamber of F_4 into that of $\text{Spin}(9)$. The number of such elements is three, given by the ratio of the orders of the Weyl groups (2.6), i.e. the subset $C \in W_{F_4}$ has one representative from each coset of $W_{\text{Spin}(9)}$. $U_{c \bullet \lambda}$ denotes the $\text{Spin}(9)$ representation with highest weight $c \bullet \lambda = c(\lambda + \rho_{F_4}) - \rho_{\text{Spin}(9)}$, with ρ the sum of fundamental weights. For F_4 , as mentioned above, there corresponds three equivalent ways of embedding $\text{Spin}(9)$ into F_4 . This implies that for each representation of F_4 , there are $\chi(F_4/\text{Spin}(9)) = 3$ irreducible representations of $\text{Spin}(9)$ generated, called the *Euler triplet*.

The consequence for eleven-dimensional supergravity is that the fields satisfy the character formula exactly for the pair $(F_4, \text{Spin}(9))$ [50]. Under the decomposition $\text{Spin}(16) \supset \text{Spin}(9)$, one of the semi-spinor representations, S^+ , stays the same, $128 = 128$, while the other, S^- , decomposes as $128' = 44 + 84$. For a highest weight $\lambda = 0$, one gets $c(\rho_{F_4}) = \rho_{SO(9)}$ the character formula is then clearly satisfied [50] as

$$\text{Id} \otimes S^+ - \text{Id} \otimes S^- = 0, \quad (2.3)$$

i.e.

$$128 - (44 + 84) = 128 - 44 - 84. \quad (2.4)$$

The Dynkin labels of the fields in the representation of $\text{Spin}(9)$ are [2000] for the graviton as a symmetric second rank tensor, [0010] for the 3rd rank antisymmetric tensor C_3 , and [1001] for the Rarita-Schwinger spinor-vector.

Remarks

1. There is a very interesting Dirac operator whose index is not zero on $\mathbb{O}P^2$. This is Kostant's cubic Dirac operator [40]

$$\mathcal{K}\xi := \sum_{a=1}^{16} \Gamma^a T^a \xi = 0, \quad (2.5)$$

where Γ^a , $a, b = 1, 2, \dots, 16$ are $2^8 \times 2^8$ gamma matrices that generate the Clifford algebra $\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$. Solutions of the Kostant equation (2.5) consists of all Euler triplets, including the supergravity multiplet [52]. The right hand side of (2.2) is the kernel of (2.5). We will deal with other Dirac operators in section 2.3.

2. The Euler characteristic of $\mathbb{O}P^2$ can be calculated as the ratio of the orders of the Weyl groups

$$\chi(\mathbb{O}P^2) = \chi(F_4/\text{Spin}(9)) = \frac{|W(F_4)|}{|W(B_4)|} = \frac{|W(F_4)|}{\mathbb{Z}_2^4 \odot S_4} = \frac{2^7 \cdot 3^2}{2^4 \cdot 4!} = 3. \quad (2.6)$$

Such a formula holds for general equal rank symmetric spaces G/H , by a classic result of Hopf and Samelson.

We now give the main theme around which this paper is centered.

Main Idea: *We interpret Ramond's triplets as arising from $\mathbb{O}P^2$ bundles with structure group F_4 over our eleven-dimensional manifold Y^{11} , on which M-theory is 'defined'.*

We have dealt with $\mathbb{O}P^2$ bundles systematically and in detail in [55], so now we proceed with the geometric interpretation of the main idea, as well as propose a geometric interpretation for the observation (2.1).

2.2 Spin(9)-structures and the M-theory fields

Before putting $\mathbb{O}P^2$ as a fiber, we start with just the space $\mathbb{O}P^2$ itself.

2.2.1 Spin(9) bundles

We start with the Spin structure on the Cayley plane.

Lemma 2.1. *$\mathbb{O}P^2$ admits a unique Spin structure.*

Over the homogeneous space $\mathbb{O}P^2 = F_4/\text{Spin}(9)$ we always have the canonical Spin(9) bundle, which we call \wp . Let $\Delta : \text{Spin}(9) \rightarrow U(16)$ be the spinor representation. We can thus form associated vector bundles with structure group $U(16)$ over $\mathbb{O}P^2$. To investigate these we should look at the K-theory of $\mathbb{O}P^2$. This has been done for general equal rank symmetric spaces G/H in [6]. The group $K^1(G/H)$ is zero, whereas $K^0(G/H)$ is a free abelian group of rank equal to the Euler number, so that $K^0(\mathbb{O}P^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Furthermore, $K^0(\mathbb{O}P^2)$ has no torsion and the Chern character map $\text{ch} : K^0(\mathbb{O}P^2) \rightarrow H^{\text{even}}(\mathbb{O}P^2; \mathbb{Q})$ is injective. Since $H^*(\mathbb{O}P^2; \mathbb{Z})$ has no torsion, K^0 is isomorphic to the cohomology of $\mathbb{O}P^2$. Therefore,

Proposition 2.2. *A complex vector bundle over $\mathbb{O}P^2$ is uniquely characterized by the classes in degrees 0, 8, and 16.*

Let $\mathfrak{R}(\text{Spin}(9))$ be the representation ring of Spin(9) and let $\beta : \mathfrak{R}(\text{Spin}(9)) \rightarrow K^0(\mathbb{O}P^2)$ be the map that assigns vector bundles over $\mathbb{O}P^2$ to representations of Spin(9), so that we have the composite map

$$\text{Spin}(9) \xrightarrow{\Delta} \mathfrak{R}(\text{Spin}(9)) \xrightarrow{\beta} K^0(\mathbb{O}P^2) \xrightarrow{\text{ch}} H^{\text{even}}(\mathbb{O}P^2; \mathbb{Q}) . \quad (2.7)$$

In fact the map β is surjective, which can be seen as follows [6]. Let s_j be the j th elementary symmetric function in the x_i^2 , where x_i , $i = 1, 2, 3, 4$, are elements of the maximal torus of Spin(9), as in [13]. Then,

using $s_2 = s_2(x_1^2, x_2^2, x_3^2, x_4^2) = \sum_{i < j} x_i x_j$ and $s_4 = s_4(x_1^2, x_2^2, x_3^2, x_4^2) = \prod_{i=1}^4 x_i^2$, the Chern character

$$\begin{aligned} \text{ch}(\beta\Delta) &= 2^4 \prod_{i=1}^4 \cosh\left(\frac{x_i}{2}\right) \\ &= \text{rk} + \frac{s_2}{6} + \text{higher terms} \\ &= 16 + u + \text{higher terms} , \end{aligned} \tag{2.8}$$

has u , the generator of $H^8(\mathbb{O}P^2; \mathbb{Z}) = \mathbb{Z}$, as a summand. Therefore we have

Proposition 2.3. *Every complex vector bundle over $\mathbb{O}P^2$ is an associated vector bundle for the $\text{Spin}(9)$ principal bundle \wp .*

2.2.2 $\text{Spin}(9)$ -structures

Let \mathfrak{f}_4 and $\mathfrak{spin}(9)$ be the Lie algebras of F_4 and $\text{Spin}(9)$, respectively. The adjoint action of F_4 is given by

$$\text{Ad}_{F_4} : F_4 \longrightarrow \text{Aut}_{\text{Lie}}(\mathfrak{f}_4). \tag{2.9}$$

Consider the restriction to $\text{Spin}(9)$

$$\text{Ad}_{F_4, \text{Spin}(9)} := \text{Ad}_{F_4}|_{\text{Spin}(9)} : \text{Spin}(9) \longrightarrow \text{Aut}_{\text{Lie}}(\mathfrak{f}_4), \tag{2.10}$$

which is given by

$$\text{Ad}_{F_4}|_{\text{Spin}(9)}(k)X = \text{Ad}_{F_4}(k)X = \text{Ad}_{\text{Spin}(9)}(k)X \in \mathfrak{spin}(9), \tag{2.11}$$

for $X \in \mathfrak{spin}(9)$ and $k \in \text{Spin}(9)$. This means that $\mathfrak{spin}(9)$ is an invariant subspace for the representation $\text{Ad}_{F_4}|_{\text{Spin}(9)}$ of $\text{Spin}(9)$ in \mathfrak{f}_4 , and there is the factor representation

$$\text{Ad}^\perp : \text{Spin}(9) \longrightarrow GL(\mathfrak{f}_4/\mathfrak{spin}(9)). \tag{2.12}$$

The sequence

$$0 \longrightarrow \mathfrak{spin}(9) \longrightarrow \mathfrak{f}_4 \longrightarrow \mathfrak{f}_4/\mathfrak{spin}(9) \longrightarrow 0 \tag{2.13}$$

is exact and $\text{Spin}(9)$ -equivariant. Consider the principal fiber bundle

$$\begin{array}{ccc} \text{Spin}(9) & \longrightarrow & F_4 \\ & & \downarrow p \\ & & F_4/\text{Spin}(9) . \end{array} \tag{2.14}$$

Using the representations (2.9) and (2.10) we form the associated bundles E_1

$$\begin{array}{ccc} \mathfrak{spin}(9) & \longrightarrow & F_4 \times_{\text{Spin}(9)} \mathfrak{f}_4/\mathfrak{spin}(9) = E_1 \\ & & \downarrow \pi_1 \\ & & F_4/\text{Spin}(9) \end{array} \tag{2.15}$$

and E_2

$$\begin{array}{ccc} \mathfrak{spin}(9) & \longrightarrow & F_4 \times_{\text{Spin}(9)} \mathfrak{spin}(9) = E_2 \\ & & \downarrow \pi_2 \\ & & F_4/\text{Spin}(9) , \end{array} \tag{2.16}$$

respectively. Then we have the following characterization of the tangent bundle of the Cayley plane.

Proposition 2.4. $T(\mathbb{O}P^2)$ is the associated vector bundle E_1 . Furthermore, $E_1 \oplus E_2$ is a trivial vector bundle.

Results for general G/K are proved in [46].

Denote by $\mathcal{F}(\mathbb{O}P^2)$ the frame bundle of the Cayley plane with structure group $SO(16)$. A $\text{Spin}(9)$ -structure is a reduction $\mathcal{R} \subset \mathcal{F}(\mathbb{O}P^2)$ of the $SO(16)$ -bundle $\mathcal{F}(\mathbb{O}P^2)$ via the homomorphism $\kappa_9 : \text{Spin}(9) \rightarrow SO(16)$. A $\text{Spin}(9)$ -structure defines certain other geometric structures [25]. In particular, it induces a 9-dimensional real, oriented Euclidean vector bundle V^9 with Spin structure

$$V^9 := \mathcal{R} \times_{\text{Spin}(9)} \mathbb{R}^9. \quad (2.17)$$

Lemma 2.5. $\mathbb{O}P^2$ admits a $\text{Spin}(9)$ -structure.

Proof. Due to the topology of $\mathbb{O}P^2$, the only nontrivial cohomology, with any coefficients, is in the top and the middle dimension (see Appendix). Then the only possible obstruction to reducing the structure group from $\text{Spin}(16)$ to $\text{Spin}(9)$ is

$$H^8 \left(\mathbb{O}P^2; \pi_{8-1} \left(\frac{\text{Spin}(16)}{\text{Spin}(9)} \right) \right). \quad (2.18)$$

From the homotopy exact sequence for the fibration

$$\text{Spin}(9) \longrightarrow \text{Spin}(16) \longrightarrow \text{Spin}(16)/\text{Spin}(9), \quad (2.19)$$

and the fact that the homotopy groups of $\text{Spin}(i)$, $i = 9, 16$ are

$$\pi_{3 \leq n \leq 15}(\text{Spin}(16)) = (\mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z} \oplus \mathbb{Z}) \quad (2.20)$$

$$\begin{aligned} \pi_{3 \leq n \leq 15}(\text{Spin}(9)) = (\mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_8, \mathbb{Z} \oplus \mathbb{Z}_2, \\ 0, \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2), \end{aligned} \quad (2.21)$$

we get that $\pi_7(\text{Spin}(16)/\text{Spin}(9)) = 0$. Therefore, there are no obstructions to reducing the structure group from $\text{Spin}(16)$ to $\text{Spin}(9)$. \square

Lemma 2.6. (Properties of V^9) **1.** *Spinors: The tangent bundle $T(\mathbb{O}P^2)$ is isomorphic to the bundle $\Delta_9(V^9)$ of real spinors of the vector bundle V^9 .*

2. *Stiefel-Whitney classes: The Stiefel-Whitney classes of $\mathbb{O}P^2$ are related to the corresponding classes of V^9 by the formula*

$$w_8(\mathbb{O}P^2) = w_4^2(V^9) + w_8(V^9). \quad (2.22)$$

3. *Pontrjagin classes:*

$$p_1(V^9) = 0 = p_3(V^9) \quad (2.23)$$

$$p_2(V^9) = -p_2(\mathbb{O}P^2) = -6u \quad (2.24)$$

$$p_4(V^9) = -\frac{1}{13}p_4(\mathbb{O}P^2) = -3u^2. \quad (2.25)$$

Proof. Part **1** follows from the definition. It is known that $\mathfrak{f}_4 = \mathfrak{so}(9) \oplus S^+$ [2] [8]. The isotropy group $\text{Spin}(9)$ acts on the tangent space $T_x \mathbb{O}P^2 = \mathfrak{f}_4/\mathfrak{spin}(9)$ as a sixteen-dimensional representation, the spinor representation Δ_9 of $\text{Spin}(9)$.

Part **2** follows from an application of the discussion in [26] for a general 16-manifold with $\text{Spin}(9)$ -structure. We just show how to get the Stiefel-Whitney classes of $\mathbb{O}P^2$. We use the Wu classes $\nu_i \in H^i(\mathbb{O}P^2; \mathbb{Z}_2)$ defined by

$$\langle \nu_i \cup u, [\mathbb{O}P^2] \rangle = \langle Sq^i u, [\mathbb{O}P^2] \rangle, \quad (2.26)$$

where Sq is the Steenrod squaring cohomology operation. Since $Sq^8 u = u^2$ then the total Wu class of $\mathbb{O}P^2$ is $\nu = 1 + u + u^2$, so that, by (2.26), the total Stiefel-Whitney class is

$$w(\mathbb{O}P^2) = Sq \nu = 1 + u + u^2. \quad (2.27)$$

For part **3** we apply theorem 2 (or corollary 3) of [25] to the case of $\mathbb{O}P^2$ so that we have the following (see Appendix for the characteristic classes of $\mathbb{O}P^2$): First $p_1(\mathbb{O}P^2) = 2p_1(V^9) = 0$.

Second, $p_2(\mathbb{O}P^2) = \frac{7}{4}(V^9) - p_2(V^9)$ so that $p_2(V^9) = -p_2(\mathbb{O}P^2)$ since $p_1(V^9)$ is zero.

Third, $p_3(\mathbb{O}P^2) = \frac{1}{8}(7p_1^3(V^9) - 12p_1(V^9)p_2(V^9) + 16p_3(V^9))$, which gives that $p_3(V^9) = 0$ since $p_2(V^9) = 0$ and $p_3(\mathbb{O}P^2) = 0$.

Fourth, $p_4(\mathbb{O}P^2) = \frac{1}{128}(35p_1^4(V^9) - 120p_1^2(V^9)p_2(V^9) + 400p_1(V^9)p_3(V^9) - 1664p_4(V^9))$, which gives $p_4(V^9) = -\frac{1}{13}p_4(\mathbb{O}P^2)$ upon using $p_1(V^9) = 0$. \square

Lemma 2.7. *The Euler class and the fourth L-polynomial of $\mathbb{O}P^2$ are given in terms of the Pontrjagin classes of V^9 as*

$$e(\mathbb{O}P^2) = \frac{p_2^2(V^9) - 4p_4(V^9)}{16} \quad (2.28)$$

$$L_4(\mathbb{O}P^2) = -\frac{1}{3^4 \cdot 5^2 \cdot 7} (19p_2^2(V^9) + 4953p_4(V^9)) \quad (2.29)$$

Proof. The formula for the Euler class follows either from substitution of the Pontrjagin classes of V^9 in terms of the Pontrjagin classes of $\mathbb{O}P^2$ in the Euler class formula of $\mathbb{O}P^2$ or directly by observing that, with $p_1(V^9) = 0$,

$$e(\mathbb{O}P^2) = \frac{1}{256}p_1^4(V^9) - \frac{1}{32}p_1^2(V^9)p_2(V^9) + \frac{1}{16}p_2^2(V^9) - \frac{1}{4}p_4(V^9) \quad (2.30)$$

gives the answer. Finally, the formula for L_4 follows by direct substitution into

$$L_4 = \frac{1}{3^4 \cdot 5^2 \cdot 7} (381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4), \quad (2.31)$$

so that

$$L_4(\mathbb{O}P^2) = -\frac{1}{3^4 \cdot 5^2 \cdot 7} (19p_2^2(V^9) + 4953p_4(V^9)) \quad (2.32)$$

\square

Remark. Using V^9 we can recover the signature of $\mathbb{O}P^2$

$$\sigma(\mathbb{O}P^2) = -\frac{1}{3} \int_{\mathbb{O}P^2} p_4(V^9) = \frac{1}{39} \int_{\mathbb{O}P^2} p_4(\mathbb{O}P^2), \quad (2.33)$$

which is related to the Euler class by $e(\mathbb{O}P^2) = 3\sigma(\mathbb{O}P^2)$.

2.2.3 Consequences for the M-theory fields

One major advantage of the introduction of an $\mathbb{O}P^2$ bundle is that in this picture the bosonic fields of M-theory, namely the metric and the C -field, can be unified.

Theorem 2.8. *The metric and the C -fields are orthogonal components of the positive spinor bundle of $\mathbb{O}P^2$.*

Proof. The spinor bundle $S^+(\mathbb{O}P^2)$ of the Cayley plane is isomorphic to

$$S^+(\mathbb{O}P^2) = S_0^2(V^9) \oplus \Lambda^3(V^9), \quad (2.34)$$

where S_0^2 denotes the space of traceless symmetric 2-tensors. This follows from an application of proposition 3 in [25] which requires the study the 16-dimensional spin representations Δ_{16}^\pm as $\text{Spin}(9)$ -representations. The element $e_1 \cdots e_{16}$ belongs to the subgroup $\widetilde{\text{Spin}}(9) \subset \text{Spin}(16)$ and acts on Δ_{16}^\pm by multiplication by (± 1) . This means that Δ_{16}^+ is an $SO(9)$ -representation, but Δ_{16}^- is a $\text{Spin}(9)$ -representation [2]. Both representations do not contain non-trivial $\text{Spin}(9)$ -invariant elements. Such an element would define a parallel spinor on $F_4/\text{Spin}(9)$ but, since the Ricci tensor of $\mathbb{O}P^2$ is not zero (see section 2.4.1), the spinor must vanish by the Lichnerowicz formula [44] $D^2 = \nabla^2 + \frac{1}{4}R_{\text{scal}}$. Then Δ_{16}^+ as a $\text{Spin}(9)$ representation is given by equation (2.34), and Δ_{16}^- is the unique irreducible $\text{Spin}(9)$ -representation of dimension 128. \square

Remarks

1. From the above we see that the Rarita-Schwinger field is given by the negative spinor bundle of $\mathbb{O}P^2$.
2. The 11-bein can also be seen from the nine-dimensional bundle in another way. It is an element of $SL(9)/\text{Spin}(9)$, which indeed has dimension 44.
3. In [39] it was shown that the bosonic degrees of freedom, g and C_3 , can be assembled into an $E_{8(+8)}$ -valued vielbein in eleven dimensions. As $E_{8(+8)}$ is the global symmetry of the two factors in the symmetry group $E_{8(+8)} \times SO(16)$, it would be interesting to see whether the discussion of the second factor here might be related to [39].

Thus we have

Theorem 2.9. *The massless fields of M-theory are encoded in the spinor bundle of $\mathbb{O}P^2$.*

Next we have the following observation

Proposition 2.10. *There is no obstruction to having sections of the $\text{Spin}(9)$ bundle on a manifold of dimension greater than or equal to 9.*

Proof. This has been observed in [26] and [34] in a different context. The real dimension of the spinor representation S is $d = 2^{\frac{m}{2}}\alpha$, where α depends on the dimension and consequently on the condition on the spinors (i.e. Majorana, Weyl), so that the maximum dimension m of the manifold M for which $d = m$ is $m = 8$. When $m > 8$ the dimensions cease to be equal anymore, $\dim S > \dim M$. The obstruction bundle is the bundle of spinors of unit norm whose fiber is $SO(d)$. As the only nontrivial homotopy group of the sphere S^{d-1} in degrees less than or equal to $d - 1$ is $\pi_{d-1}(S^{d-1}) = \mathbb{Z}$, the primary- and only- obstruction lies in $H^d(M^m; \mathbb{Z})$. For $n \geq 9$ one has $d > m$, so that the obstruction is zero. \square

Remark. We can use the twisted geometric Dirac operator introduced in [43] to give another interpretation of the the Euler triplet in M-theory. Since $\mathfrak{o}P^2$ is Spin, the identity representation of F_4 is the index of the the Dirac operator on $\mathfrak{o}P^2$ twisted by the homogeneous vector bundle induced by the representation of Spin(9). Calling this representation \mathcal{V} and consider the representations S_+^* and S_-^* , dual to half-Spin representations S^+ and S^- , respectively. Applying [43], we have the twisted Dirac operator

$$D_{S(\mathfrak{o}P^2) \otimes \mathcal{V}} : L^2(F_4 \times_{\text{Spin}(9)} (S_+^* \otimes \mathcal{V})) \longrightarrow L^2(F_4 \times_{\text{Spin}(9)} (S_-^* \otimes \mathcal{V})) , \quad (2.35)$$

whose index is

$$\text{Index } D_{S(\mathfrak{o}P^2) \otimes \mathcal{V}} = \text{Id}(F_4) . \quad (2.36)$$

2.3 Supersymmetry

We have seen that supersymmetry is created from bundles on $\mathbb{O}P^2$. More precisely, this is really due to parallel spinors on \mathbb{R}^9 . In fact, this can be seen from another angle. There is a supersymmetric structure inside of V^9 , which makes \mathfrak{f}_4 into a Lie superalgebra. The connection comes from the relation between real Killing spinors on a space and the parallel spinors on the cone over that space [9]. Let us see how this works, following [23]. The eight-sphere S^8 with the standard round metric g has a Spin bundle $S(S^8)$ on which there is an action of the Clifford bundle $\mathcal{C}\ell(TS^8)$ and a Spin(8) invariant inner product. A Killing spinor over S^8 is a nonzero section ϵ of $S(S^8)$ which satisfies, for all vector fields X ,

$$\nabla_X \epsilon = \lambda X \cdot \epsilon, \quad (2.37)$$

with Killing constant $\lambda \in \mathbb{R}$. In local coordinates, using $\lambda = \frac{1}{2}$, this is

$$(\nabla_\mu - \frac{1}{2}\gamma_\mu)\epsilon = 0. \quad (2.38)$$

The cone on S^8 is $\mathcal{C}S^8 = \mathbb{R}^9 \setminus \{0\}$. The metric $dr^2 + r^2g$, however, can be extended to the origin, so that we can take the cone to be \mathbb{R}^9 . Thus

$$\begin{aligned} \text{Parallel spinors on } \mathbb{R}^9 &\iff \text{Real Killing spinors on } S^8 \\ \nabla_\mu \hat{\epsilon} = 0 &\iff (\nabla_\mu - \frac{1}{2}\gamma_\mu)\epsilon = 0 . \end{aligned} \quad (2.39)$$

The observation in [23] is that this decomposition, written as $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1$, has the interpretation in terms of Killing superalgebras on S^8 : $\mathfrak{l}_0 = \mathfrak{so}(9)$ is the Lie algebra of isometries of S^8 and $\mathfrak{l}_1 = S^+$ is the space of Killing spinors on S^8 . The latter comes, via the cone construction, from real Killing spinors on the cone \mathbb{R}^9 . Hence

$$\mathfrak{f}_4 = \{\text{Even isometries on } S^8\} \oplus \{\text{Odd isometries on } S^8\} , \quad (2.40)$$

and the Lie brackets for the super Lie algebra are satisfied [23]. Schematically (abusing notation of fiber vs. bundle), we have

$$\begin{array}{ccccc} \underbrace{\text{Spin}(9) - \text{structures}}_{V_9} & \longleftarrow & \underbrace{\text{Killing spinors}}_{S^8} & \longrightarrow & \underbrace{\text{parallel spinors}}_{\mathcal{C}S^8} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{O}P^2 & & \end{array} \quad (2.41)$$

From this and the earlier discussion we therefore have

Proposition 2.11. *\mathfrak{f}_4 is the Lie superalgebra of a sphere inside V^9 . Hence the unification of the fields in M-theory and their supersymmetry can be seen from the eight-sphere over $\mathbb{O}P^2$.*

We can give another interpretation to the Euler triplet in terms of spinors. We have seen in the Remark containing equation (2.36) that the Euler triplet can be interpreted as an index of a twisted Dirac operator. The kernel of the operator (2.35) is the space of harmonic spinors, which is the desired representation up to sign [43]. Therefore, we get another characterization of the supergravity multiplet.

Proposition 2.12. *The identity representation of F_4 encoding the supergravity multiplet is the space of twisted harmonic spinors on $\mathfrak{O}P^2$.*

Comparison to generation of supersymmetry from lattices. Next we discuss the relation, similarities and differences between the above process of generating fermions and supersymmetry and the one through which the various closed superstring theories are derived starting from the closed bosonic string [16]. The spectrum of the bosonic string contains no fermions and so these are generated on a lattice in internal space. In [16] the following procedure was created:

- (1) Seek an internal symmetry group G containing the little group $\text{Spin}(8)$. This is achieved by a torus compactification T/Λ_G , with Λ_G the root lattice of a simply-laced group G of rank 8.
- (2) Declare the diagonal subgroup $SO(8)_{\text{diag}} \subset SO(8) \times \text{Spin}(8)$ as the new transverse group. This implies that the spinor representations of $\text{Spin}(8)$ describe fermionic states.
- (3) Extend $SO(8)_{\text{diag}}$ to the full Lorentz group $SO(1, 9)_{\text{diag}}$.
- (4) Impose the supersymmetry requirement that a consistent truncation on the spectrum of the bosonic string be performed. This requires a regular embedding so that the root lattice $\Lambda_{\text{Spin}(8)}$ is contained in Λ_G .

The only simply-laced groups which contain $\text{Spin}(8)$ as a subgroup in a regular embedding are E_6 , E_7 and E_8 . Requiring the rank to be 8 then singles out $G = E_8 \times E_8$. Then [16]:

- (i) the choice $G_L = G_R = E_8 \times E_8$ for the groups in the left and right sector gives the two type II string theories;
- (ii) the same choice with a truncation on the left-moving sector gives the $E_8 \times E_8$ heterotic string;
- (iii) the choice $G_L = E_8 \times E_8$, $G_R = \text{Spin}(32)/\mathbb{Z}_2$ together with a truncation on the left-moving sector gives the $\text{Spin}(32)/\mathbb{Z}_2$ heterotic string theory.

Now let us compare the similarities and the differences of our case with the above formalism of [16]. We record this in the following remarks.

Remarks

1. The M-theory case is geometric and involves nontrivial topology. This is in contrast to the torus in a vertex-operator-like construction in the string case.
2. F_4 is not simply-laced and hence cannot be involved in the internal torus construction.
3. In both cases, the fermions are generated from the internal space. However, in [16], fermionic states are generated from bosonic states. In fact, in our case, the whole massless spectrum of eleven-dimensional supergravity is generated from the two Spin bundles in dimension sixteen. This method of generating fermions is very different from the string formalism of generating fermions from torus compactification.
4. The signature $\sigma(M^{4k})$ of an oriented $4k$ -dimensional manifold M^{4k} is an invariant of the manifold. Moreover, the signature of $-M^{4k}$, which is M^{4k} with the orientation reversed, is equal to the negative of

the signature of M^{4k} : $\sigma(-M^{4k}) = -\sigma(M^{4k})$. Since $\sigma(\mathbb{O}P^2) \neq 0$, this means that there is no orientation-reversing homeomorphism $f : \mathbb{O}P^2 \rightarrow \mathbb{O}P^2$ such that $f_*[\mathbb{O}P^2] = -[\mathbb{O}P^2]$. The implication is, in particular, that we cannot impose any such involution on the fermions.

5. The construction in M-theory using F_4 involves the Spin bundle of $\mathbb{O}P^2$. This means that in twenty-seven dimensions the theory will have fermions. This is a major difference from the bosonic string case, which has no fermions in its spectrum. How can this be compatible with the bosonic string and with the classification of supersymmetry in general? In relation to the bosonic string, it could be that there is an involution that kills the fermions in a way similar to what happens to the C -field in going from M-theory to the heterotic string, or from the conjectural bosonic M-theory in [31] to bosonic string theory. Let us now consider the second part of the question related to the classification of supersymmetry. The action in twenty-seven dimensions might involve fermions, and so the question is whether this will/can be supersymmetric. That is something to be investigated. However, for now we can say that being supersymmetric does not contradict the no-go theorems in supersymmetry as those involve the Lorentz condition. The sixteen-dimensional internal space can be taken to have either all time or all space signature, i.e. $(16, 0)$ or $(0, 16)$, respectively. We then get for the signature (t, s) of the 27-dimensional space

$$(1, 10) + (0, 16) = (1, 26) \quad (2.42)$$

$$(1, 10) + (16, 0) = (17, 10). \quad (2.43)$$

The first one obviously wildly violates the no-go theorems but the second does not as $t - s = 7$. Note that a version of eleven-dimensional M-theory with $s - t = 7$ was constructed by Hull [32]. While supersymmetry seems mathematically admissible, it is far from obvious what to make physically of so many such time directions. We do not address this here.

2.4 Relating Y^{11} and M^{27}

2.4.1 geometric consequences

We start with the Riemannian geometry of $\mathbb{O}P^2$. Consider the following three subsets of \mathbb{O}^3

$$U_1 = \{1\} \times \mathbb{O} \times \mathbb{O}, \quad U_2 = \mathbb{O} \times \{1\} \times \mathbb{O}, \quad U_3 = \mathbb{O} \times \mathbb{O} \times \{1\}, \quad (2.44)$$

and form the union $\mathcal{U} := U_1 \cup U_2 \cup U_3$. Define the following relation \sim on \mathbb{O}^3 :

$$[a, b, c] \sim [d, e, f] \iff \text{there exists } \lambda \in \mathbb{O} - \{0\} \text{ such that } a = d\lambda, b = e\lambda, c = f\lambda. \quad (2.45)$$

The relation \sim on \mathcal{U} is an equivalence relation [5]. The Cayley projective plane is the set of equivalence classes of \mathcal{U} by the equivalence relation \sim ,

$$\mathbb{O}P^2 = \mathcal{U} / \sim. \quad (2.46)$$

Keeping in mind $\mathbb{O} \cong \mathbb{R}^8$, an atlas on $\mathbb{O}P^2$ can be taken to be $(U_i / \sim, \phi_i)$, $i = 1, 2, 3$, where the homeomorphisms ϕ_i are given by

$$\begin{aligned} \phi_1 &: U_1 / \sim \longrightarrow \mathbb{R}^{16}, & \phi_1([a, b, c]) &= (b, c), \\ \phi_2 &: U_2 / \sim \longrightarrow \mathbb{R}^{16}, & \phi_2([a, b, c]) &= (a, c), \\ \phi_3 &: U_3 / \sim \longrightarrow \mathbb{R}^{16}, & \phi_3([a, b, c]) &= (a, b). \end{aligned} \quad (2.47)$$

The transition functions $\phi_i \circ \phi_j^{-1} : \mathbb{R}^{16} \rightarrow \mathbb{R}^{16}$

$$\begin{aligned}\phi_1 \circ \phi_2^{-1}(a, b) &= (a^{-1}, ba^{-1}) = \phi_2 \circ \phi_1^{-1}(a, b), \\ \phi_1 \circ \phi_3^{-1}(a, b) &= (ba^{-1}, a^{-1}) = \phi_3 \circ \phi_1^{-1}(a, b), \\ \phi_2 \circ \phi_3^{-1}(a, b) &= (b^{-1}, ab^{-1}) = \phi_3 \circ \phi_2^{-1}(a, b)\end{aligned}\tag{2.48}$$

are diffeomorphisms and hence we get a smooth 16-dimensional manifold structure for $\mathbb{O}P^2$ [29].

The metric on $\mathbb{O}P^2$ can be obtained from the metrics on the charts which are compatible with respect to transition maps. The metric, with (u, v) coordinate functions, is [29]

$$ds^2 = \frac{|du|^2(1 + |v|^2) + |dv|^2(1 + |u|^2) - 2\text{Re}[(u\bar{v})(dv d\bar{u})]}{(1 + |u|^2 + |v|^2)^2}.\tag{2.49}$$

In terms of a coordinate frame $\{e_1, \dots, e_8, f_1, \dots, f_8\}$ where $e_i = \partial_i$ and $f_i = \partial_{i+8}$ for $1 \leq i \leq 8$, the unmixed components of the metric are

$$\begin{aligned}g(e_i, e_j) &= \delta_{ij} \frac{1 + |v|^2}{(1 + |u|^2 + |v|^2)^2}, \\ g(f_i, f_j) &= \delta_{ij} \frac{1 + |u|^2}{(1 + |u|^2 + |v|^2)^2}.\end{aligned}\tag{2.50}$$

The mixed components, in terms of the standard orthonormal basis $\{x_1, \dots, x_8\}$ of \mathbb{O} are

$$g(e_i, f_j) = g(f_i, e_j) = -\frac{\langle (u\bar{v})x_j, x_i \rangle}{(1 + |u|^2 + |v|^2)^2}.\tag{2.51}$$

Using the identity

$$R_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda}{}^\sigma = \Gamma_{\mu\lambda;\nu}^\sigma - \Gamma_{\nu\lambda;\mu}^\sigma = \frac{1}{2} [g_{\nu\lambda;\mu\sigma} + g_{\mu\sigma;\nu\lambda} - g_{\mu\lambda;\nu\sigma} - g_{\nu\sigma;\mu\lambda}],\tag{2.52}$$

the only non-vanishing components of the Riemann tensor are [29]

$$\begin{aligned}R(e_i, e_j, e_i, e_j) &= -R(e_i, e_j, e_j, e_i) = 4, \\ R(f_i, f_j, f_i, f_j) &= -R(f_i, f_j, f_j, f_i) = 4, \\ R(e_i, e_j, f_k, f_l) &= R(f_k, f_l, e_i, e_j) = -\langle x_i \bar{x}_l, x_j \bar{x}_k \rangle + \langle x_j \bar{x}_l, x_i \bar{x}_k \rangle, \\ R(e_i, f_j, e_k, f_l) &= R(f_i, e_j, f_k, e_l) = \langle x_i \bar{x}_j, x_k \bar{x}_l \rangle, \\ R(f_i, e_j, e_l, f_k) &= -\langle x_i \bar{x}_j, x_k \bar{x}_l \rangle.\end{aligned}\tag{2.53}$$

It can now be easily seen that both the Ricci curvature tensor $R_{\mu\nu}$ and the Ricci scalar R are both positive.

Taking M^{27} to be the total space of an $\mathbb{O}P^2$ bundle over Y^{11} then the Ricci curvatures of the two spaces are related. In particular, since $\mathbb{O}P^2$ is a compact Riemannian manifold which has a metric of positive Ricci curvature on which the Lie group F_4 acts by isometries, and the base Y^{11} is a compact manifold, it follows from O'Neill's formulae for submersions (see [11]) that

Proposition 2.13. *If the base Y^{11} admits a metric of positive Ricci curvature, then so does the 27-dimensional space.*

This is shown by taking a certain metric on M^{27} with totally geodesic fibers ([11]) and then shrinking the $\mathbb{O}P^2$ fibers à la Kaluza-Klein. This is a specific case of the $\mathbb{O}P^2$ analog of Proposition 3.6 in [58].

2.5 Structures on M^{27}

The cohomology of $\mathbb{O}P^2$ is $H^*(\mathbb{O}P^2; C) = C[x]/x^3$, $|x| = \deg x = 8$, as an algebra.

Remarks

1. Note that a priori the characteristic of C should divide the order of the Weyl group of F_4 . Since $|W(F_4)| = 2^7 \cdot 3^2$ then the candidate primes are 2 and 3 only. We have seen that among these two numbers only the prime 3 gives a nontrivial Serre fibration.
2. Note that the primes 2 and 3 are also the torsion primes of F_4 . It is not the case in general that the torsion primes for G are exactly the same primes that appear in the factorization of $|W(G)|$.

The total space of an $\mathbb{H}P^2$ bundle over a Spin manifold is again a Spin manifold. However, the same property is not automatically true for total spaces of $\mathbb{O}P^2$ bundles over $BO\langle 8 \rangle$ -manifolds. The reason is that while the tangent bundle T along the fibers of the universal bundle

$$\mathbb{O}P^2 = F_4/\text{Spin}(9) \longrightarrow B\text{Spin}(9) \longrightarrow BF_4 \quad (2.54)$$

has a Spin structure — since $H^i(B\text{Spin}(9)) = 0$ for $i = 1, 2, 3$ — it has no $BO\langle 8 \rangle$ structure. This can be explained as follows, using [37]. The complementary roots of $i : \text{Spin}(9) \hookrightarrow F_4$ are the 16 roots $\frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4)$, where x_i denote the standard linear forms on $\mathfrak{so}(9)$. Using Borel-Hirzebruch methods [13], the total Pontrjagin class $p(T) \in H^*(B\text{Spin}(9); \mathbb{Q})$ is given by the product $\frac{1}{4} \prod (\pm x_1 \pm x_2 \pm x_3 \pm x_4)$, so that the first Pontrjagin class is

$$p_1(T) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) \in H^4(B\text{Spin}(9); \mathbb{Q}). \quad (2.55)$$

This is of course invariant under the Weyl group of $\text{Spin}(9)$. However, it is also invariant under $W(F_4)$, and hence belongs to $H^4(BF_4; \mathbb{Q}) = \mathbb{Q}$ as well. This shows that $p_1(T)$ can be considered as coming from the universal space for $\text{Spin}(9)$ or F_4 .

Proposition 2.14. *If Y^{11} admits a String structure then so does M^{27} provided that there is no contribution from the degree four class from BF_4 .*

Proof. We have the $\mathbb{O}P^2$ bundle over Y^{11} with total space M^{27}

$$\begin{array}{ccc} M^{27} & \xrightarrow{\tilde{f}} & B\text{Spin}(9) \\ \pi \downarrow & & \downarrow Bi \\ Y^{11} & \xrightarrow{f} & BF_4 \end{array} \quad (2.56)$$

which gives the decomposition $TM^{27} = \pi^*TY^{11} \oplus \tilde{f}^*T$, and so the tangential Pontrjagin class is

$$p_1(M^{27}) = \pi^*(p_1(Y^{11}) + f^*p_1(T)). \quad (2.57)$$

In the case Y^{11} is a 3-connected $BO\langle 8 \rangle$ -manifold, we have that $H^4(Y^{11}; \mathbb{Z})$ is free and $\pi^* : H^4(Y^{11}; \mathbb{Z}) \rightarrow H^4(M^{27}; \mathbb{Z})$ is an isomorphism. Thus M^{27} is also a $BO\langle 8 \rangle$ -manifold if and only if $f^*\bar{x}_4 = 0 \in H^4(Y^{11}; \mathbb{Z})$, where $\bar{x}_4 \in H^4(BF_4; \mathbb{Z})$ is the generator. Therefore we have shown that M^{27} is String if and only if G_4 in M-theory gets no contribution from BF_4 . \square

Remarks

1. The quantization condition for the field strength G_4 in M-theory is known [61]. Since this field does not seem to get a contribution from a class in BF_4 , the condition in Proposition 2.14 seems reasonable. In some sense we could view the presence of such a degree four class as an anomaly which we have just cured. Alternatively, one can discover that this is not as serious as it might seem— see the more complete discussion in section 3.2.

2. We connect the above discussion back to cobordism groups. While there is no transfer map from $\Omega_{11}^{(8)}(BF_4)$ to $\Omega_{27}^{(8)}$, there is a transfer map after killing \bar{x}_4 [37]. Denoting by ${}^b BF_4\langle\bar{x}_4\rangle$ the corresponding classifying space that fibers over BF_4 , killing \bar{x}_4 is done by pulling back the path fibration $PK(\mathbb{Z}, 4) \rightarrow K(\mathbb{Z}, 4)$ with a map $\bar{x}_4 : BF_4 \rightarrow K(\mathbb{Z}, 4)$ realizing \bar{x}_4 . The corresponding transfer map is $\Omega_{11}^{(8)}(BF_4\langle\bar{x}_4\rangle) \rightarrow \Omega_{27}^{(8)}$.

Next, for the higher structures we consider String and Fivebrane structures [56] [57]. We have

Proposition 2.15. 1. *In order for M^{27} to admit a Fivebrane structure, the second Pontrjagin class of Y^{11} should be the negative of that of $\mathbb{O}P^2$, i.e. $p_2(TY^{11}) = -p_2(T\mathbb{O}P^2) = -6u$.*

2. *$\hat{A}(M^{27}) = 0$, irrespective of whether or not the \hat{A} -genus of Y^{11} is zero.*

3. *The Witten genus $\Phi_W(M^{27}) = 0$.*

4. *The elliptic genus $\Phi_{\text{ell}}(M^{27}) = 0$.*

Proof. For part (1) note that if Y^{11} admits a Fivebrane structure then M^{27} does not necessarily admit such a structure. This is because the obstruction to having a Fivebrane structure is $\frac{1}{6}p_2$ [57] but we know that $\frac{1}{6}p_2(\mathbb{O}P^2) = u \neq 0$. However, we can choose Y^{11} appropriately so that it conspires with $\mathbb{O}P^2$ to cancel the obstruction and lead to a Fivebrane structure for M^{27} . Noting that the tangent bundles are related as $TM^{27} = TY^{11} \oplus T\mathbb{O}P^2$, then considering the degree eight part of the formula (see [47])

$$p(E \oplus F) = \sum p(E)p(F) \quad \text{mod } 2\text{-torsion.} \quad (2.58)$$

we get for our spaces

$$p_2(TY^{11} \oplus T\mathbb{O}P^2) = p_1(TY^{11})p_1(T\mathbb{O}P^2) + p_2(TY^{11}) + p_2(T\mathbb{O}P^2) \quad \text{mod } 2\text{-torsion.} \quad (2.59)$$

Since we have $p_1(T\mathbb{O}P^2) = 0$, then requiring that $p_2(TM^{27}) = 0$ leads to the constraint that $p_2(TY^{11}) + p_2(T\mathbb{O}P^2) = 0$ modulo 2-torsion.

For part (2) we use the multiplicative property of the \hat{A} -genus for Spin fiber bundles to get

$$\hat{A}(M^{27}) = \hat{A}(Y^{11})\hat{A}(\mathbb{O}P^2). \quad (2.60)$$

Since the \hat{A} -genus of $\mathbb{O}P^2$ is zero then the result follows.

For part (3) we use a result of Ochanine [49]. Taking the total space M^{27} and the base Y^{11} to be closed oriented manifolds, and since the fiber $\mathbb{O}P^2$ is a Spin manifold and the structure group F_4 of the bundle is compact, then the multiplicative property of the genus can be applied

$$\Phi_W(M^{27}) = \Phi_W(\mathbb{O}P^2)\Phi_W(Y^{11}). \quad (2.61)$$

Now since we proved in [55] that $\Phi_W(\mathbb{O}P^2) = 0$, it follows immediately that $\Phi(M^{27})$ is zero regardless of whether or not $\Phi_W(Y^{11})$ vanishes. Even more, $\Phi_W(Y^{11})$ is zero because Y^{11} is odd-dimensional. ^c

^bThis is the analog of the String group when $G = \text{Spin}$, in the sense that it is the 3-connected cover.

^c However, see the case when Y^{11} is a circle bundle at the end of this section.

For part (4) we use the fact that the fiber is Spin and the structure group F_4 is compact and connected so we can apply the multiplicative property of the elliptic genus [49]

$$\Phi_{\text{ell}}(M^{27}) = \Phi_{\text{ell}}(Y^{11})\Phi_{\text{ell}}(\mathbb{O}P^2). \quad (2.62)$$

In this case the genus for the fiber is not zero (see [55]) but the elliptic genus of Y^{11} is zero, again because of dimension. Therefore $\Phi_{\text{ell}}(M^{27}) = 0$. \square

Ochanine genera. There is another description of the Ochanine k -invariant [38], which we will use to make a connection to invariants appearing in M-theory.

Proposition 2.16. *1. The Ochanine invariant of a ten-dimensional closed Spin manifold X^{10} is equal to the mod 2 index of the Dirac operator twisted with the virtual bundle $TX^{10} - 2$.*

Proof. The family index theorem says that for E a real bundle in $KO^0(X^{10})$ an invariant $e \in \mathbb{Z}_2$ was defined by Atiyah and Singer [7] by $\langle E, [X^{10}]_{KO} \rangle = e\eta^2\mu \in KO_{10}$, which turns out to be the mod 2 index of the Dirac operator D_E of X^{10} twisted by the virtual bundle E ,

$$e = \dim_{\mathbb{C}} \ker(D_E) \pmod{2}. \quad (2.63)$$

Applying [37], the k -invariant of X^{10} is the coefficient of q in the expression $f(q)^{-8}\Phi_{\text{och}} \in KO_{10}[[q]]$, where

$$f(q) := \sum_{n \geq 1} q^{\binom{n}{2}} = 1 + q + q^3 + q^6 + \dots, \quad (2.64)$$

since $\varepsilon/q = f(q^8) \pmod{2} = f(q)^8 \pmod{2}$. We find the coefficient of q in the expansions. We have

$$f(q)^{-8} = (1 + q + \dots)^{-8} = 1 - 8q + \dots. \quad (2.65)$$

The expansion for $\theta(q)$ takes the form

$$\theta(q) = \left(\frac{1-q}{1-q^2} \right) \left(\frac{1-q^3}{1-q^4} \right) \dots = 1 - q + \dots, \quad (2.66)$$

so that $\theta(q)^{-10} = 1 + 10q + \dots$. The expansion of $\Theta_q(E)$ is

$$\begin{aligned} \Theta_q(E) &= \Lambda_{-q}(E) \otimes S_{q^2}(E) + \dots \\ &= \left(\sum_{k \geq 0} (-q)^k \Lambda^k(E) \right) \otimes \left(\sum_{k \geq 0} (q^2)^k S^k(E) \right) \\ &= 1 - qE + \dots. \end{aligned} \quad (2.67)$$

Putting the expressions (2.65), (2.66), (2.67) together we get

$$\begin{aligned} f(q)^{-8}\theta(q)^{-10}\Theta_q(TX^{10}) &= (1 - 8q + \dots)(1 + 10q + \dots)(1 - qTX^{10} + \dots) \\ &= 1 + (2 - TX^{10})q + \dots. \end{aligned} \quad (2.68)$$

Extracting the coefficient of q we get the desired result.

Note that there is another way of obtaining this which makes use of the grading for Φ_{och} . Instead of looking at θ_q and Θ_q separately, we can look at the coefficient of q in the Ochanine genus $\Phi_{\text{och}}(X^{10})$. This is

$$\Phi_{\text{och}}^1 = \langle -\Pi_1(TX^{10}), [X^{10}]_{KO} \rangle \in KO_{10} = \mathbb{Z}_2, \quad (2.69)$$

where Π_1 is the first KO-Pontrjagin class (defined in [4]), which is equal to $\Lambda^1(TX^{10} - 10) = TX^{10} - 10$. Substituting in (2.69) we get

$$\Phi_{\text{och}}^1 = \langle -(TX^{10} - 10), [X^{10}]_{KO} \rangle, \quad (2.70)$$

which agrees with the product $\theta_q^{-10} \Theta_q(TX^{10}) = 1 + (10 - TX^{10})q + \dots$. \square

Remarks

1. Note that, interestingly, the bundle we get is the Rarita-Schwinger bundle with the dilatino and the spinor ghosts, as the Rarita-Schwinger field Ψ which leads to gauge invariance is a section of $SX^{10} \otimes (TX^{10} - 2\mathcal{O})$, where \mathcal{O} is the trivial line bundle. The (mod 2) index I_{RS} of the corresponding Dirac operator D_{RS} appears in the phase of the partition function [19] through the phase of the Pfaffian

$$Pf(D_{RS}) = (-1)^{I_{RS}/2} |Pf(D_{RS})|. \quad (2.71)$$

What is remarkable is that the ‘quantum’ Rarita-Schwinger operator appears directly in this formulation.

2. In [19] the main focus was the dependence of the partition function on the degree four class a coming from the E_8 gauge theory, but the contribution from I_{RS} was also given. The main example discussed in [19] is $X^{10} = \mathbb{H}P^2 \times T^2$. Using the property

$$k(M^8 \times S^1 \times S^1) = \sigma(M^8) \pmod{2}, \quad (2.72)$$

we can indeed see that the Ochanine k -invariant in this case is not zero. With T^2 taken as the product of two circles with nontrivial Spin structures we have

$$k(S^1 \times S^1 \times \mathbb{H}P^2) = \sigma(\mathbb{H}P^2) \pmod{2}, \quad (2.73)$$

which is equal to 1, since $\sigma(\mathbb{H}P^2) = 1$.

3. In defining the elliptically refined partition function in M-theory and type IIA string theory, a real-oriented elliptic cohomology theory appears [41]. This is $EO(2)$, the fixed point, with respect to the formal inverse, of the theory $E\mathbb{R}(2)$, the real version of Morava theory $E(2)$, which has two generators v_1 and v_2 . The orientation in this theory is shown to be given by w_4 [41]. It was also shown that when $w_4(X^{10}) = 0$, X^{10} has an $EO(2)$ -orientation class $[X^{10}]_{EO(2)_{10}} \in EO(2)_{10}(X^{10})$, and for $x \in E^0(X^{10})$, the refined mod 2 index in this theory is

$$j(x) = \langle x, [X^{10}]_{EO(2)} \rangle \in EO(2)_{10} = \mathbb{Z}_2[v_1^3 v_2^{-1}]. \quad (2.74)$$

3 Terms in the Lifted Action

In this section we consider possible terms in the lifted action up in twenty-seven dimensions. We first consider geometric expressions involving a distinguished 8-form, called the Cayley 8-form, in section 3.1. Then in section 3.2 we consider torsion classes and their effect on the M-theory partition function.

3.1 The Cayley 8-form

In section 2.3 we discussed the question of whether the higher-dimensional ‘theory’ in our case is supersymmetric. In any case holonomy would give us a handle on whatever differential forms end up appearing. The holonomy group of $\mathbb{O}P^2$ is $\text{Spin}(9)$ and there is in fact a $\text{Spin}(9)$ -invariant 8-form that generalizes the Kähler 2-form for $\mathbb{C}P^2$ and the fundamental or Cayley 4-form on $\mathbb{H}P^2$ [15]. The $\text{Spin}(9)$ representation $\Lambda^8(\Delta_9) = \Lambda^8(\mathbb{R}^{16})$ contains a unique 8-form which is invariant under the action of $\text{Spin}(9)$. Note that $\mathbb{O}P^2$ does not admit an almost complex structure [13] nor an almost quaternionic structure [10].

The explicit expression for the 8-form is given in terms of the cross product of vectors $V_i = (0, e_i)$, $e_i \in \mathbb{O}$, $i = 1, \dots, 8$, in the tangent plane $\mathbb{O} \oplus \mathbb{O}$ to $\mathbb{O}P^2$ by [15] [14]

$$\omega_8(V_1, V_2, \dots, V_8) = \frac{1}{8!} \sum_{\sigma \in \Sigma_8} \varepsilon(\sigma) [(V_{\sigma(1)} \times V_{\sigma(2)}) (V_{\sigma(3)} \times V_{\sigma(4)})] [(V_{\sigma(5)} \times V_{\sigma(6)}) (V_{\sigma(7)} \times V_{\sigma(8)})]. \quad (3.1)$$

Note that ω_8 is nonzero, real, takes the value 1 on $\mathbb{O}P^2$, and reduces to a product of two fundamental Cayley calibration 4-forms ϕ upon restriction to $\mathbb{O}P^1$ [14]

$$\omega_8(e_1, e_2, \dots, e_8) = \frac{1}{35} \sum_{P_8^4} \phi(e_1, e_2, e_3, e_4) \cdot \phi(e_5, e_6, e_7, e_8). \quad (3.2)$$

In fact there is another expression for the Cayley 8-form which corresponds to the integral generator of the cohomology ring of $\mathbb{O}P^2$. This is described as follows [1]. Let u_i and v_i , $i = 1, \dots, 8$, be 1-forms on $T\mathbb{O}P^2$ satisfying

$$\begin{aligned} v_i(e_j, 0) &= \delta_{ij} & v_i(0, e_j) &= 0 \\ u_i(e_j, 0) &= 0 & u_i(0, e_j) &= \delta_{ij}. \end{aligned} \quad (3.3)$$

Various 2-forms can be formed, such as $\omega_{IJ} = v_I \wedge v_J$ and $\eta_{KL} = u_K \wedge u_L$, where I, J, K, L are various combinations of pairs of i and j . The $\text{Spin}(9)$ -invariant 8-form ω_8 is the sum of eight 8-forms $\omega_8 = \sum_{i=1}^8 \omega_8^i$, where ω_8^i are built out of wedge products of the v_i , u_j , ω_{IJ} and η_{KL} . More precisely,

$$\omega_8^1 = -14(v_1 \wedge \dots \wedge v_8 - u_1 \wedge \dots \wedge u_8), \quad (3.4)$$

and ω_8^m , $m = 2, \dots, 8$ are quartic expressions in ω_{IJ} and η_{KL} . The action of the Lie algebra $\mathfrak{spin}(9)$ on any 8-form φ is

$$(\alpha\varphi)(X_1, \dots, X_8) = \sum_{i=1}^8 \varphi(X_1, \dots, \alpha X_i, \dots, X_8), \quad (3.5)$$

for $\alpha \in \mathfrak{spin}(9)$ and $X_1, \dots, X_8 \in T\mathbb{O}P^2$. The 8-form ω_8 satisfies $\alpha\omega_8 = 0$, so that it is $\text{Spin}(9)$ -invariant. The advantage of this approach is that the identification with the cohomology generator is possible and transparent, even though it take some work to write down the form itself. Set

$$\omega_{16} = v_1 \wedge v_2 \wedge \dots \wedge v_8 \wedge u_1 \wedge u_2 \wedge \dots \wedge u_8, \quad (3.6)$$

the analog of the epsilon symbol whose integral is the volume form of $\mathbb{O}P^2$. The wedge product of ω_8 with itself gives

$$\omega_8 \wedge \omega_8 = 1848 \omega_{16}. \quad (3.7)$$

Set $\mathcal{J}_8 = \frac{60}{\pi^4} \omega_8$. We will need the volume of $\mathbb{O}P^2$. For sake of this calculation we can take $\mathbb{O}P^2$ to be

$$\{\text{lines in } \mathbb{O}^3 \cong \mathbb{R}^{24}\} = \frac{S^{23}}{S^7}. \quad (3.8)$$

Now using the fact that the volume of the sphere S^{d-1} of unit radius and geodesic length 2π is $2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$, we get

$$\text{vol}(\mathbb{O}P^2) = \frac{2\pi^{11}}{\Gamma(11)} \frac{\Gamma(3)}{2\pi^3} = \frac{3! \pi^8}{11!}, \quad (3.9)$$

with normalization of geodesic length π . Now evaluating the wedge product of the 8-form with itself over $\mathbb{O}P^2$, and using (3.7), gives

$$\begin{aligned} \int_{\mathbb{O}P^2} \mathcal{J}_8 \wedge \mathcal{J}_8 &= \int_{\mathbb{O}P^2} \frac{2^4 \cdot 3^2 \cdot 5^2}{\pi^8} \omega_8 \wedge \omega_8 \\ &= \frac{(2^4 \cdot 3^2 \cdot 5^2)(2^3 \cdot 3 \cdot 7 \cdot 11)}{\pi^8} \int_{\mathbb{O}P^2} \omega_{16} = 1, \end{aligned} \quad (3.10)$$

since $1848 = 2^3 \cdot 3 \cdot 7 \cdot 11$. Let $H_{DR}^*(\mathbb{O}P^2)$ denote the de Rham cohomology ring of $\mathbb{O}P^2$. Let

$$r : H_{DR}^*(\mathbb{O}P^2) \longrightarrow H^*(\mathbb{O}P^2; \mathbb{R}) \quad (3.11)$$

be the de Rham isomorphism, and

$$j : H^*(\mathbb{O}P^2; \mathbb{Z}) \longrightarrow H^*(\mathbb{O}P^2; \mathbb{R}) \quad (3.12)$$

be the homomorphism induced by the natural homomorphism from \mathbb{Z} to \mathbb{R} . Finally, the structure of the cohomology ring $H^*(\mathbb{O}P^2; \mathbb{Z}) = \mathbb{Z}[u]/u^3$, which implies that the generator of degree 16 is the square of the generator of degree 8, gives

$$r([\mathcal{J}_8]) = \pm j(u). \quad (3.13)$$

Therefore the class $[\mathcal{J}_8]$ of the closed differential form \mathcal{J}_8 corresponds to the integral generator u of $H^8(\mathbb{O}P^2; \mathbb{Z})$.

Note that the 8-form has the following properties:

- (1) The 8-form defines a unique parallel form on $\mathbb{O}P^2$.
- (2) Since the signature of $\mathbb{O}P^2$ is positive, then the 8-form is self-dual.

Remarks

1. At the rational level we can thus use ω_8 to build a $\text{Spin}(9)$ -invariant degree sixteen expression

$$\rho_{16}^{\mathbb{R}} = \omega_8 \wedge \omega_8 \quad (3.14)$$

that we integrate and insert as part of the action as $\int_{\mathbb{O}P^2} \rho_{16}^{\mathbb{R}}$.

2. Assume that there are fields \mathcal{F}_8 and \mathcal{F}_{16} in the 27-dimensional ‘theory’ with potentials \mathcal{C}_7 and \mathcal{C}_{15} . In the dimensional reduction on $\mathbb{O}P^2$ to eleven dimensions, a natural $\text{Spin}(9)$ -invariant ansatz for the fields may be taken, at the rational level, to be

$$\mathcal{F}_8 = \omega_8, \quad \mathcal{F}_{16} = \omega_8 \wedge \omega_8, \quad (3.15)$$

and similar expressions at the integral level in terms of \mathcal{J}_8 . Note that since ω_{16} is essentially the volume form, then such an ansatz is the analog of the Freund-Rubin ansatz [24] in the reduction of eleven-dimensional supergravity to lower dimensions.

3.2 Torsion classes and effect on the M-theory partition function

In subtle situations the fields in the physical theory can be torsion classes in cohomology. We consider terms in the action coming from BF_4 or from the fiber $\mathbb{O}P^2$. We will show that torsion classes from BF_4 are compatible with the description in [19] of the phase of the M-theory partition function.

3.2.1 Classes from BF_4

1. \mathbb{Z}_2 coefficients: The cohomology ring of BF_4 with coefficients in \mathbb{Z}_2 is given from by the polynomial ring [12]

$$H^*(BF_4; \mathbb{Z}_2) = \mathbb{Z}_2 [x_4, Sq^2 x_4, Sq^3 x_4, x_{16}, Sq^8 x_{16}], \quad (3.16)$$

where x_4 and x_{16} are polynomial generators of degree 4 and 16, respectively. From the structure of the cohomology ring (3.16) we see that we can pull back classes from BF_4 and that these are in fact compatible with the fields of M-theory. In particular, there is a degree four class x_4 , as in all Lie groups of dimension greater than or equal to three, which could be matched with the field strength G_4 in M-theory. In fact, since *any* degree four class can be the characteristic class a_{E_8} of an E_8 bundle, then a class pulled back from F_4 can certainly be at the same time a class of some E_8 bundle. Hence an F_4 class is possible in the shifted quantization condition

$$[G_4] - \frac{\lambda}{2} = a_{E_8} \in H^4(Y^{11}; \mathbb{Z}), \quad (3.17)$$

discovered in [61].

The higher degree classes are also relevant. We also have the degree six and the seven generators $Sq^2 x_4$ and $Sq^3 x_4$, respectively, which, when nonzero, would appear in the phase of the partition function. The comparison of M-theory on Y^{11} with type IIA string theory on a ten-manifold X^{10} involves the bilinear form [19]

$$\mu(a, b) = \int_{X^{10}} a \cup Sq^2 b, \quad (3.18)$$

for $a, b \in H^4(X^{10}; \mathbb{Z})$. This can be viewed [19] as a torsion pairing

$$\begin{aligned} T : H_{\text{tor}}^4(X^{10}; \mathbb{Z}) \times H_{\text{tor}}^7(X^{10}; \mathbb{Z}) &\longrightarrow U(1) \\ (a, Sq^3 b) &\longmapsto \int_{X^{10}} a \cup Sq^2 c, \end{aligned} \quad (3.19)$$

where $Sq^3 b = \beta(Sq^2 c) = Sq^1 Sq^2 c = Sq^3 c$. In our case a and b can be $f^* x_4$. Thus we have

Proposition 3.1. *\mathbb{Z}_2 classes from BF_4 are compatible with the M-theory partition function, i.e. they produce no anomalies and they do not change the value of the partition function.*

2. \mathbb{Z}_3 coefficients: If we restrict to low degrees, say ≤ 16 , then we have the truncated polynomial

$$H^*(BF_4; \mathbb{Z}_3) \cong \mathbb{Z}_3[x_4, x_8] + \Lambda(x_9). \quad (3.20)$$

Now the main observation is that the class x_9 , being $\beta P_3^1 x_4$, is the same as the class required to be cancelled in theorem ???. If we kill this class then we are left with only the degree four and the degree eight classes x_4 and x_8 . Since x_8 is $P_3^1 x_4$, then this $\mathbb{Z}_3[x_4, P_3^1 x_4]$ is also compatible with the mod 3 description of the anomalies in M-theory described in [54]. Therefore,

Proposition 3.2. *\mathbb{Z}_3 classes from BF_4 are compatible with the partition function of M-theory once the anomaly $P_3^1 x_4$ is cancelled.*

3.2.2 Classes from $\mathbb{O}P^2$

Recall that we have introduced fields \mathcal{F}_8 and \mathcal{F}_{16} with corresponding potentials \mathcal{C}_7 and \mathcal{C}_{15} , respectively (see (3.15)). Assuming that the 27-dimensional ‘theory’ indeed has such fields, we consider some consequences in this section. We emphasize that we do not have enough knowledge about the dynamics (if and when it exists) in 27 dimensions so we will concentrate on the topology. We will concentrate on the first field, because of the cohomology of $\mathbb{O}P^2$, i.e. that the second would probably be a ‘composite’ of the first.

Imposing conventional Dirac quantization on the field \mathcal{C}_7 gives that these fields are classified topologically by a class $x \in H^8(\mathbb{O}P^2; \mathbb{Z})$, so that x is represented in de Rham cohomology by $\frac{\mathcal{F}_8}{2\pi}$,

$$x = \left[\frac{\mathcal{F}_8}{2\pi} \right]. \quad (3.21)$$

In analogy to the case in string theory [63] and M-theory [61] [62], we consider the construction of the partition function corresponding to \mathcal{C}_7 . This is done in terms of a theta function on $T = H^8(\mathbb{O}P^2; U(1))$. However, since $\mathbb{O}P^2$ has no torsion in cohomology, then T will be the torus

$$T = H^8(\mathbb{O}P^2; \mathbb{R}) / H^8(\mathbb{O}P^2; \mathbb{Z}). \quad (3.22)$$

Furthermore, the construction requires a function

$$\Omega : H^8(\mathbb{O}P^2; \mathbb{Z}) \longrightarrow \mathbb{Z}_2, \quad (3.23)$$

obeying the law

$$\Omega(x + y) = \Omega(x)\Omega(y)(-1)^{x \cdot y}, \quad (3.24)$$

where $x \cdot y$ is the intersection pairing $\int_{\mathbb{O}P^2} x \cup y$ on $\mathbb{O}P^2$. The function Ω enters into the determination of the line bundle \mathcal{L} on T . The partition function of the \mathcal{C}_7 field will then be a holomorphic section of \mathcal{L} .

The signature of $\mathbb{O}P^2$, which has dimension 16, is by definition the signature of the quadratic form

$$\begin{aligned} H^8(\mathbb{O}P^2; \mathbb{Q}) &\longrightarrow \mathbb{Q} \\ v &\longmapsto \langle v^2, [\mathbb{O}P^2] \rangle, \end{aligned} \quad (3.25)$$

whose value is 1.

The intersection form. For a manifold M^{2n} of dimension $2n$, the universal coefficient theorem implies that

$$H_n(M^{2n}; \mathbb{R}) \cong H_n(M^{2n}) \otimes \mathbb{R} \cong (H_n(M^{2n})/T_n) \otimes \mathbb{R}. \quad (3.26)$$

Torsion elements do not affect the intersection number: if α_n, β_n are torsion elements so that $r\alpha_n, s\beta_n \in H_n(M^{2n}; \mathbb{R})$, then

$$\langle r\alpha_n, s\beta_n \rangle = rs \langle \alpha_n, \beta_n \rangle, \quad (3.27)$$

so that the intersection forms over \mathbb{R} and \mathbb{Z} have the same matrix. Then $H_n(M^{2n}; \mathbb{R})$ has a basis in which the intersection form has integer coefficients. Since the cup product is anti-commutative then the intersection form is symmetric for even n and antisymmetric for odd n . The intersection form of $\mathbb{O}P^2$ is not even. This can be seen in two ways. First that the signature of $\mathbb{O}P^2$, which is the signature of the intersection matrix of the middle cohomology of $\mathbb{O}P^2$, is not zero. Second, the Steenrod operation $Sq^4 k$ does not decompose in the similar way that Sq^{4k+2} does. In the latter case, the Adem relation $Sq^{4k+2} = Sq^2 Sq^{4k} + Sq^1 Sq^{4k} Sq^1$ implies that $x_{4k+2}^2 = Sq^{4k+2} x_{4k+2} = 0$.

Now we look at mod 2 and integral bilinear forms. We have

Proposition 3.3. 1. *The bilinear form*

$$\begin{aligned} H^8(\mathbb{O}P^2; \mathbb{Z}_2) \times H^8(\mathbb{O}P^2; \mathbb{Z}_2) &\longrightarrow \mathbb{Z}_2 \\ (a_8, a_8) &\longmapsto \int_{\mathbb{O}P^2} a_8 \cup a_8 \end{aligned} \quad (3.28)$$

is given by $\int_{\mathbb{O}P^2} a_8 \cup w_8$.

2. *The bilinear form over \mathbb{Z}*

$$H^8(\mathbb{O}P^2; \mathbb{Z}) \times H^8(\mathbb{O}P^2; \mathbb{Z}) \longrightarrow \mathbb{Z} \quad (3.29)$$

is an odd \mathbb{Z} -form.

Proof. Consider the first part. Since $w_8^2 = p_4 \bmod 2$ and $w_{16} = 3u^2 = e \bmod 2$, then the total Stiefel-Whitney class of $\mathbb{O}P^2$ is $w = 1 + u + u^2$, with coefficients of u reduced mod 2 [13] (see equation (2.27)). The fact that the first seven Stiefel-Whitney classes of $\mathbb{O}P^2$ vanish implies that the Wu class $\nu(\mathbb{O}P^2)$ reduces to the element $w_8(\mathbb{O}P^2) \in H^8(\mathbb{O}P^2; \mathbb{Z}_2)$ [33]. Consequently, the Stiefel-Whitney class $w_8(\mathbb{O}P^2)$ is characterized by the condition [25]

$$y_8 \cup y_8 = y_8 \cup w_8(\mathbb{O}P^2) \text{ for any } y_8 \in H^8(\mathbb{O}P^2; \mathbb{Z}_2). \quad (3.30)$$

Next consider the second part. In [25] it was shown that, for a compact manifold M^{16} admitting a $\text{Spin}(9)$ -structure, the quadratic form

$$H^8(M^{16}; \mathbb{Z})/\text{Tor} \times H^8(M^{16}; \mathbb{Z})/\text{Tor} \longrightarrow H^{16}(M^{16}; \mathbb{Z}) \quad (3.31)$$

is an even \mathbb{Z} -form if and only if $w_8(M^{16}) = 0$. Since $\mathbb{O}P^2$ has no torsion in cohomology, $H^{16}(\mathbb{O}P^2; \mathbb{Z}) = \mathbb{Z}$, and $w_8(\mathbb{O}P^2)$ is nonzero, then the result follows immediately. \square

In fact, we know that the value of the intersection form is given by the signature, which is 1.

3.3 Further terms and compatibility with other theories

3.3.1 Kinetic terms

We have not so far included any kinetic terms in the discussion. The main reason is that we do not know the nature of the resulting ‘theory’ and whether it will have such terms. If we take the proposal in [31], there are difficulties with the Einstein-Hilbert, i.e. the gravitational kinetic, term because the obvious choice does not give the correct term in bosonic string theory in twenty-six dimensions upon dimensional reduction, but is off by a factor of 125/121. This is also linked with difficulties of finding coset symmetries [36] [42]. Thus we exclude the gravitational terms from the discussion. We go back to some of this in section 3.3.3. To some limited extent, we do consider the kinetic term for the M-theory C -field provided this field lifts and provided that such a term does in fact appear.

Assuming a kinetic term for G_4 , then the EOM would be rationally

$$d *_ {27} G_4 = \frac{1}{2} G_4 \wedge G_4 \wedge Z_{16} + I_8 \wedge Z_{16}, \quad (3.32)$$

where $*_{27}$ is the Hodge duality operator in 27 dimensions. The right hand side is a degree 24 differential form, whose class is of the form

$$\Theta_{24}^{\mathbb{R}} := \left[\frac{1}{2} G_4 \wedge G_4 + I_8 \right] \wedge Z_{16}. \quad (3.33)$$

As we have argued earlier, a term such as Z_{16} can only be a composite, i.e. a square of degree eight expressions, due to the cohomology of $\mathbb{O}P^2$. We are interested in the integral lift of that degree 24 expression. The term in brackets in (3.33) has an integral lift given by the class Θ_8 , defined in [18], as $[\Theta_8(a)]_{\mathbb{R}} = \frac{1}{2}a_{\mathbb{R}}(a_{\mathbb{R}} - \lambda_{\mathbb{R}}) + 30\hat{A}_8$. The integral lift of Z_{16} is just u^2 where u is the generator of $H^8(\mathbb{O}P^2; \mathbb{Z})$. Thus we have

Proposition 3.4. *The integral lift of $\Theta_{24}^{\mathbb{R}}$ is given by*

$$[\Theta_{24}] = [\Theta_8] \cup u^2 \quad (3.34)$$

The study of this class, and further refinements thereof, could be useful.

Remark. Having $*_{27}G_4$ and $[\Theta_{24}]$ signals the appearance of 21-branes in the 27-dimensional theory. Requirement of decoupling of this brane from the membrane, so that a well-defined partition function can be constructed, gives that the class $[\Theta_{24}]$ be trivial in cohomology, so that the fields are cohomologically trivial on the brane. One obvious way to ensure this is to require triviality of $[\Theta_8]$. If we do not require this then we can find some other way to do this. We do not just set u to zero. But we can do something when reducing coefficients. Let P_5^1 be the Steenrod reduced power operation $P_5^1 : H^k(\mathbb{O}P^2; \mathbb{Z}_5) \longrightarrow H^{k+8}(\mathbb{O}P^2; \mathbb{Z}_5)$. Let \bar{u} be the generator u with coefficients reduced mod 5. In this case, for $k = 8$, the action of P_5^1 is given by multiplication with $5L_2$, where L_2 is the degree 8 term in the L -genus [30].

$$P_5^1 \bar{u} = \frac{1}{9}(7p_2 - p_1^2)\bar{u} = -2p_2\bar{u} = -2\bar{u}^2. \quad (3.35)$$

This implies the following.

1. We can make $[\Theta_{24}]$ zero by imposing the condition $P_5^1 \bar{u} = 0$. This is analogous to the mod 3 case in [54].
2. For each homeomorphism $\phi : \mathbb{O}P^2 \rightarrow \mathbb{O}P^2$, $\phi^* \bar{u} = \bar{u}$ [13]. Hence \bar{u} is invariant under continuous deformations of $\mathbb{O}P^2$.

3.3.2 Compatibility with ten-dimensional superstring theories

We have looked at the proposed ‘theory’ in twenty-seven dimensions in relation to M-theory in eleven dimensions. The question will now be whether the structures we discussed are compatible with other known theories. Given that the 27-dimensional ‘theory’ is proposed in such a way that it is by construction compatible with M-theory (as we know it) then, since all five superstring theories in ten dimensions are obtained from M-theory via dimensional reduction and/or dualities, the 27-dimensional construction is compatible with these superstring theories. We will actually reduce the $F_4 - \mathbb{O}P^2$ -bundle to ten dimensions along the M-theory circle and check this explicitly.

We consider the $\mathbb{O}P^2$ bundle M^{27} with structure group F_4 . The transition functions on Y^{11} , with patches U_i and U_j , will be

$$g_{ij} : U_i \cap U_j \longrightarrow \text{Diff}(\mathbb{O}P^2), \quad (3.36)$$

are $\text{Diff}(\mathbb{O}P^2)$ -valued (Diff^+ if orientation-preserving). If we take Y^{11} to be the product $X^{10} \times S^1$ and view

the circle as the interval $[0, 1]$ with the ends glued together then we can form the diagram

$$\begin{array}{ccc}
\mathbb{O}P^2 & \xrightarrow{=} & \mathbb{O}P^2 \\
\downarrow & & \downarrow \\
M^{27} & \xrightarrow{\quad} & \pi^* M^{27} \\
\downarrow & & \downarrow \\
X^{10} \times S^1 & \xleftarrow{\pi} & X^{10} \times [0, 1] .
\end{array} \tag{3.37}$$

The bundle $\pi^* M^{27}$ is isomorphic to a bundle $\xi^{26} \times [0, 1]$ over $X^{10} \times [0, 1]$. Gluing at $[0, 1]$ we get a map from X^{10} to $\text{Aut}(\xi^{26})$, the automorphism group of the bundle ξ . Therefore,

Proposition 3.5. *From a bundle M^{27} over $X^{10} \times S^1$ we get*

1. *a bundle $\xi^{26} \rightarrow X^{10}$ with fiber $\mathbb{O}P^2$ and structure group F_4 , and*
2. *a gauge group element of ξ^{26} , i.e. a map $X^{10} \rightarrow \text{Aut}(\xi^{26})$.*

If the bundle is trivial then the automorphisms of ξ^{26} will be the automorphisms of the fiber, i.e. F_4 . A map from X^{10} to F_4 might then be regarded as a classifying map for based loop bundles, since $B\Omega F_4 = F_4$. Thus, in this special case, we have an F_4 bundle and an ΩF_4 bundle over X^{10} . This is analogous to the case of E_8 [53].

The diffeomorphism group above is very large and is not easy to work with. Instead we will invoke a condition that is familiar from Kaluza-Klein theory, namely to assume that the original bundle comes from a principal F_4 -bundle

$$\begin{array}{ccc}
F_4 & \longrightarrow & P \\
& & \downarrow \\
& & Y^{11},
\end{array} \tag{3.38}$$

so that we effectively consider the reduction of the structure group $\text{Diff}^+(\mathbb{O}P^2)$ to the subgroup F_4 , the isometry group of the $\mathbb{O}P^2$ fiber. This is analogous to the case when Y^{11} itself is taken as the total space of a circle bundle over X^{10} . A priori the structure group is $\text{Diff}^+(S^1)$, in which the transition functions are valued. Restricting to $U(1) \subset \text{Diff}^+(S^1)$, we get a principal circle bundle $U(1) \rightarrow Y^{11} \rightarrow X^{10}$. In fact, in this case, the reduction is always possible and no condition is required. Now we are presented with a situation which is analogous to having an E_8 bundle [61] in eleven dimensions that we are asking to reduce to ten dimensions. The result, analogously to the E_8 case [3] [45], is

$$\begin{array}{ccccc}
F_4 & \longrightarrow & P & & \\
& & \downarrow & & \\
S^1 & \longrightarrow & Y^{11} & \implies & LF_4 \longrightarrow Q \\
& & \downarrow & & \downarrow \\
& & X^{10} & & X^{10} .
\end{array} \tag{3.39}$$

The homotopy type of F_4 is identical to the homotopy type of E_8 in degrees less than eleven, and so rationally $F_4 \sim S^3$, $\Omega F_4 \sim S^4$, so that $LF_4 \sim S^3 \times S^4$. Thus, at the rational level, we expect a degree three and a degree four class from the LF_4 bundle. At the integral level, since $F_4 \sim K(\mathbb{Z}, 3)$, then

$$LF_4 \sim K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 4), \quad \deg < 11. \tag{3.40}$$

This can be shown as follows. We have LF_4 bundles which are classified by maps to BLF_4 . The sequence $\Omega X \rightarrow LX \rightarrow X$ for $X = BF_4$ gives

$$F_4 \longrightarrow LBF_4 \longrightarrow BF_4 . \quad (3.41)$$

Since F_4 is connected, then LBF_4 and BLF_4 are homotopy equivalent. We can then replace LBF_4 with BLF_4 in (3.42). Since 2 and 3 are the only torsion primes for F_4 , then for $p \geq 5$ the sequence

$$F_4 \longrightarrow BLF_4 \begin{array}{c} \xrightarrow{\text{ev}} \\ \xleftarrow{s} \end{array} BF_4 \quad (3.42)$$

splits on mod p cohomology, so that

$$H^*(BLF_4; \mathbb{Z}_p) \cong H^*(BF_4; \mathbb{Z}_p) \otimes H^*(F_4; \mathbb{Z}_p), \quad p \geq 5 , \quad (3.43)$$

as algebras. At the torsion primes we use the Serre spectral sequence corresponding to the sequence (3.42). From (3.16) we see for $p = 2$ that in degrees ≤ 15 ,

$$H^*(BF_4; \mathbb{Z}_2) = \mathbb{Z}_2 [x_4, Sq^2 x_4, Sq^3 x_4] . \quad (3.44)$$

The differential d acting on x_4 is zero because of the section s in (3.42). From (3.44), for $p = 2$, and from (3.20), for $p = 3$, we see that all the generators are connected by cohomology operations, Sq^i and P^j , respectively. Thus, since $x_{i>4} = \mathcal{O}x_4$, for some cohomology operation \mathcal{O} , then all the differentials are zero. Thus the spectral sequence collapses and the fibration is a product.

The LF_4 bundle over X^{10} is therefore a $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$ bundle. The first factor, $K(\mathbb{Z}, 2)$ gives the NS field H_3 and the second factor, $K(\mathbb{Z}, 3)$ gives the RR field F_4 in ten dimensions. Hence at the topological level, compatibility of F_4 with ten-dimensional type IIA is reduced to that of E_8 , which follows from [19] [3] [45]. The compatibility with type IIB, and hence with F-theory, also follows from T-duality as for the E_8 case [22]. Therefore, we can give the following statement.

Proposition 3.6. *Consider the $\mathbb{O}P^2$ bundle over Y^{11} with structure group reduced to F_4 as above. Then*

1. *The reduction of the F_4 bundle on the circle in Y^{11} leads to an LF_4 bundle over X^{10} .*
2. *At the topological level, the $\mathbb{O}P^2$ bundle, with the above assumptions, is compatible with type II string theory.*

3.3.3 Compatibility with the bosonic string

The question is whether the 27-dimensional structure is compatible with the bosonic string theory in twenty-six dimensions, on X^{26} . We have addressed some aspects of this in section 2.3 in relation to fermions and supersymmetry, and so we consider other aspects in this section. The form fields we have introduced, including G_4 from M-theory, are all of dimensions that are multiples of 4. Since the bosonic string spectrum does not involve G_4 and the action does not obviously get the topological terms that we introduced, then the relation between M^{27} and X^{26} , if a dimensional reduction, could be a one-dimensional orbifold,^d i.e. S^1/\mathbb{Z}_2 , where we assume a \mathbb{Z}_2 parity on all form fields of degrees of the form $4k$ in such a way that they disappear in the same way that G_4 gets killed in going from M-theory to the heterotic theory and also from

^dAlternatively, the relation between the twenty-seven - and the twenty-six-dimensional theories could be more involved such as in the case of heterotic/type II duality.

the bosonic theory in [31] to twenty-six dimensions. Thus, the forms coming from the $\mathbb{O}P^2$ bundles can be made compatible with bosonic string theory.

One difficulty with the proposal in [31] was raised in [42], which is that the action does not support a coset symmetry that would include the bosonic string theory. This was also observed in [36]. The question is whether our proposal can evade these objections. In [42] the reduction was on tori, but ours is a coset space with large and sparse homotopy cells. In [36] the analysis was based on assumptions, such as Lorentz symmetry, that we do not know whether they hold for the higher-dimensional case, and the search was made based on the classification of simple Lie algebras. It is possible that the higher structures will not be entirely described by such classical notions (although of course we used some of these notions in our own discussion). Furthermore, in both [42] and [36] gravity was involved. The Einstein-Hilbert term in twenty-seven dimensions does not give the correct term in twenty-six dimensions [31], and this is related to the lack of coset symmetry structure [42] mentioned above. We have not included the gravitational kinetic terms in our discussion, mainly for this reason, but also because there is a possibility that the theory will not be of the usual form. This was also raised in [42]. It is possible that the theory will be nonlocal or topological. We cannot answer this in any definitive way here.

Thus, given the discussion about supersymmetry at the end of section 2.3 and the above discussion, it would be desirable to find a compatibility diagram of the schematic form

$$\begin{array}{ccc}
 M^{27} & \xrightarrow{\quad ? \quad} & X^{26} \\
 \text{\(\mathbb{O}P^2\) reduction} \downarrow & & \downarrow \text{Lattice reduction} \\
 Y^{11} & \xrightarrow[\text{reduction}]{S^1 \text{ or } S^1/\mathbb{Z}_2} & M^{10} .
 \end{array} \tag{3.45}$$

This requires further investigation but we have not immediately seen an obstruction for this to hold.

Acknowledgements

The author thanks the American Institute of Mathematics for hospitality and the “Algebraic Topology and Physics” SQuaRE program participants for very useful discussions. The author would like to thank the Hausdorff Institute for Mathematics in Bonn for hospitality and the organizers of the “Geometry and Physics” Trimester Program at HIM for the inspiring atmosphere during the writing of this paper. Special thanks are due Pierre Ramond for helpful remarks and encouragement and to Arthur Greenspoon for many useful editorial suggestions.

4 Appendix: Some Properties of $\mathbb{O}P^2$

In this appendix we summarize the topological properties of the Cayley plane $\mathbb{O}P^2$ which are useful for proving some of the results in the text.

1. **Betti numbers:** The only nonzero Betti numbers are $b_0 = 1$, $b_8 = 1$, $b_{16} = 1$.
2. **Integral cohomology:** The cohomology ring is

$$H^*(\mathbb{O}P^2; \mathbb{Z}) = \mathbb{Z}[u]/u^3, \tag{4.1}$$

where $u \in H^8(\mathbb{O}P^2; \mathbb{Z})$ is the canonical 8-dimensional generator coming from S^8 . Thus $H^0\mathbb{Z} = H^8\mathbb{Z} = H^{16}\mathbb{Z} = \mathbb{Z}$ and $H^i = 0$ otherwise. Note that there is no torsion in cohomology. Consider the last Hopf map $S^7 \longrightarrow S^{15} \xrightarrow{f} S^8$. The spheres S^7 and S^8 are oriented, so that generators $a \in H^7(S^7; \mathbb{Z}) = \mathbb{Z}$ and $b \in H^8(S^8; \mathbb{Z})$ can be specified. The mapping cone $\mathcal{C}(f)$ is $\mathbb{O}P^2$. The exactness of the cohomology long exact sequence corresponding to f gives the isomorphisms

$$\begin{aligned} \iota & : H^{15}(S^{15}; \mathbb{Z}) \xrightarrow{\cong} H^{16}(\mathbb{O}P^2; \mathbb{Z}) \\ j^* & : H^8(\mathbb{O}P^2; \mathbb{Z}) \xrightarrow{\cong} H^8(S^8; \mathbb{Z}) . \end{aligned} \quad (4.2)$$

Let $a' = \iota(a) \in H^{16}(\mathbb{O}P^2; \mathbb{Z})$, and let $u \in H^8(\mathbb{O}P^2; \mathbb{Z})$ be the unique element such that $j^*(u) = b$. Since $H^{16}(\mathbb{O}P^2; \mathbb{Z}) = \mathbb{Z}$ then there exists a unique integer $H(f)$, the *Hopf invariant*, such that $u \cup u = H(f)a'$. It is a classical result that this is equal to one. Therefore $a' = u^2$. This justifies the above claim about the cohomology of $\mathbb{O}P^2$.

3. **Euler class:** Let u be a generator of $H^8(\mathbb{O}P^2; \mathbb{Z})$. The Euler class of $\mathbb{O}P^2$ is $e = \pm 3u^2$.
4. **Pontrjagin classes:** The total tangential Pontrjagin class is given by [13]

$$p(T\mathbb{O}P^2) = 1 + 6u + 39u^2, \quad (4.3)$$

so that the nonzero Pontrjagin classes are $p_2 = 6u$, $p_4 = 39u^2$. Choosing that orientation which is defined by u^2 , the non-vanishing Pontrjagin numbers are $p_2^2[\mathbb{O}P^2] = 36$, $p_4[\mathbb{O}P^2] = 39$.

References

- [1] K. Abe and M. Matsubara, *Invariant forms on the exceptional symmetric spaces FII and EIII*, Transformation group theory (Taejŏn, 1996), 3–16, Korea Adv. Inst. Sci. Tech., Taejŏn, 1996.
- [2] J. F. Adams, *Lectures on exceptional Lie groups*, University of Chicago Press, Chicago, IL, 1996.
- [3] A. Adams and J. Evslin, *The loop group of E_8 and K -theory from 11d*, J. High Energy Phys. **0302** (2003) 029, [[arXiv:hep-th/0203218](#)].
- [4] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson, *SU-cobordism, KO-characteristic numbers, and the Kervaire invariant*, Ann. of Math. **(2) 83** (1966) 54–67.
- [5] H. Aslaksen, *Restricted homogeneous coordinates for the Cayley projective plane*, Geom. Dedicata **40** (1991), no. 2, 245–250.
- [6] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III pp. 7–38, AMS, Providence, R.I., 1961.
- [7] M. F. Atiyah and I. M. Singer, *The index of elliptic operators. V*, Ann. of Math. **(2) 93** (1971) 139–149.
- [8] J. Baez, *The octonions*, Bull. Amer. Math. Soc. **39** (2002) 145–205. Erratum ibid. **42** (2005) 213, [[arXiv:math/0105155](#)] [[math.RA](#)].
- [9] C. Bär, *Real Killing spinors and holonomy*, Commun. Math. Phys. **154** (1993) 509–521.
- [10] C. Bär, *Elliptic symbols*, Math. Nachr. **201** (1999), 7–35.

- [11] A. L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin, 1987.
- [12] A. Borel, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. J. Math. **76** (1954) 273–342.
- [13] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces. I*, Amer. J. Math. **80** (1958) 458–538.
- [14] C. Brada and F. Pecaut-Tison, *Géométrie du plan projectif des octaves de Cayley*, Geom. Dedicata **23** (1987) no. 2, 131–154.
- [15] R. B. Brown and A. Gray, *Riemannian manifolds with holonomy group $Spin(9)$* , Differential geometry (in honor of Kentaro Yano), eds. S. Kobayashi et al., pp. 41–59. Kinokuniya, Tokyo, 1972.
- [16] A. Casher, F. Englert, H. Nicolai, and A. Taormina, *Consistent superstrings as solutions of the $D = 26$ bosonic string theory*, Phys. Lett. **B162** (1985) 121.
- [17] E. Cremmer, B. Julia, and J. Scherk, *Supergravity theory in eleven-dimensions*, Phys. Lett. **B76** (1978) 409–412.
- [18] E. Diaconescu, D. S. Freed, and G. Moore, *The M -theory 3-form and E_8 gauge theory*, Elliptic cohomology, 44–88, London Math. Soc. Lecture Note Ser., 342, Cambridge Univ. Press, Cambridge, 2007, [[arXiv:hep-th/0312069](#)].
- [19] E. Diaconescu, G. Moore, and E. Witten, *E_8 gauge theory, and a derivation of K -Theory from M -Theory*, Adv. Theor. Math. Phys. **6** (2003) 1031, [[arXiv:hep-th/0005090](#)].
- [20] M. J. Duff, *$E_8 \times SO(16)$ symmetry of $d = 11$ supergravity?*, in Quantum Field Theory and Quantum Statistics, eds. I. A. Batalin, C. Isham and G. A. Vilkovisky, Adam Hilger, Bristol, UK 1986.
- [21] M. J. Duff, *M -theory (the theory formerly known as strings)*, Int. J. Mod. Phys. **A11** (1996) 5623–5642, [[arXiv:hep-th/9608117v3](#)].
- [22] J. Evslin, *From E_8 to F via T* , J. High Energy Phys. **0408** (2004) 021, [[arXiv:hep-th/0311235](#)].
- [23] J. Figueroa-O’Farrill, *A geometric construction of the exceptional Lie algebras F_4 and E_8* , [[arXiv:0706.2829v](#)] [[math.DG](#)].
- [24] P. G. O. Freund and M. A. Rubin, *Dynamics of dimensional reduction*, Phys. Lett. **B97** (1980) 233–235.
- [25] T. Friedrich, *Weak $Spin(9)$ -structures on 16-dimensional Riemannian manifolds*, Asian J. Math. **5** (2001) 129–160, [[arXiv:math/9912112](#)] [[math.DG](#)].
- [26] A. Gray and P. Green, *Sphere transitive structures and the triality automorphism*, Pac. J. Math. **34** (1970), 83–96.
- [27] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory Vol. 2*, second edition, Cambridge University Press, Cambridge, 1988.
- [28] B. Gross, B. Kostant, P. Ramond, and S. Sternberg, *The Weyl character formula, the half-spin representations, and equal rank subgroups* Proc. Natl. Acad. Sci. USA **95** (1998), no. 15, 8441–8442, [[arXiv:math/9808133](#)] [[math.RT](#)].

- [29] R. Held, I. Stavrov, and B. Van Kote, *(Semi-)Riemannian geometry of (para-)octonionic projective planes*, [[arXiv:math/0702631](#)] [[math.DG](#)].
- [30] F. Hirzebruch, *On Steenrod's reduced powers, the index of inertia, and the Todd genus*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953) 951–956.
- [31] G. Horowitz and L. Susskind, *Bosonic M-theory*, J. Math. Phys. **42**(2001) 3152, [[arXiv:hep-th/0012037](#)].
- [32] C. M. Hull, *Duality and the signature of space-time*, J. High Energy Phys. **9811** (1998) 017, [[arXiv:hep-th/9807127](#)].
- [33] D. Husemoller, *Fibre bundles*, Third edition, Springer-Verlag, New York, 1994.
- [34] C. J. Isham, C. N. Pope, and N. P. Warner, *Nowhere-vanishing spinors and triality rotations in 8-manifolds*, Class. Quant. Gravity **5** (1988), no. 10, 1297–1311.
- [35] B. Julia, *Group disintegrations*, in Superspace and supergravity, S.W. Hawking and M. Roček (eds.), Cambridge Univ. Press, Cambridge, UK, 1981.
- [36] A. Keurentjes, *The group theory of oxidation*, Nucl. Phys. **B658** (2003) 303–347, [[arXiv:hep-th/0210178](#)].
- [37] S. Klaus, *Brown-Kervaire invariants*, PhD thesis, University of Mainz, Shaker Verlag, Aachen, 1995.
- [38] S. Klaus, *The Ochanine k-invariant is a Brown-Kervaire invariant*, Topology **36** (1997), no. 1, 257–270.
- [39] K. Koepsell, H. Nicolai, and H. Samtleben, *An exceptional geometry for d=11 supergravity?*, Class. Quant. Grav. **17** (2000) 3689–3702, [[arXiv:hep-th/0006034](#)].
- [40] B. Kostant, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. J. **100** (1999), no. 3, 447–501.
- [41] I. Kriz and H. Sati, *M Theory, type IIA superstrings, and elliptic cohomology*, Adv. Theor. Math. Phys. **8** (2004) 345, [[arXiv:hep-th/0404013](#)].
- [42] N. D. Lambert and P. C. West, *Coset symmetries in dimensionally reduced bosonic string theory*, Nucl. Phys. **B615** (2001) 117–132, [[arXiv:hep-th/0107209](#)].
- [43] G. Landweber, *Harmonic spinors on homogeneous spaces*, Represent. Theory **4** (2000) 466–473, [[arXiv:math/0005056v1](#)] [[math.DG](#)].
- [44] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris **257** (1963) 7–9.
- [45] V. Mathai and H. Sati, *Some relations between twisted K-theory and E_8 gauge theory*, J. High Energy Phys. **0403** (2004) 016, [[arXiv:hep-th/0312033](#)].
- [46] P. W. Michor, *Gauge theory for fiber bundles*, Bibliopolis, Napoli, 1991.
- [47] J. Milnor and J. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, NJ, 1974.
- [48] H. Nicolai, *D = 11 supergravity with local SO(16) invariance*, Phys. Lett. **B187** (1987) 316.

- [49] S. Ochanine, *Genres elliptiques équivariants*, in Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986), 107–122, Lecture Notes in Math., 1326, Springer, Berlin, 1988.
- [50] T. Pengpan and P. Ramond, *M(ysterious) Patterns in SO(9)*, Phys. Rept. **315** (1999) 137-152, [[arXiv:hep-th/9808190](#)].
- [51] P. Ramond, *Boson-fermion confusion: The string path to supersymmetry*, Nucl. Phys. Proc. Suppl. **101** (2001) 45-53, [[arXiv:hep-th/0102012](#)].
- [52] P. Ramond, *Algebraic dreams*, Meeting on Strings and Gravity: Tying the Forces Together, Brussels, Belgium, 19-21 Oct 2001, [[arXiv:hep-th/0112261](#)].
- [53] H. Sati, *Loop group of E_8 and targets for spacetime*, Mod. Phys. Lett. **A 24** (2009) 25, [[arXiv:hep-th/070123](#)].
- [54] H. Sati, *An approach to anomalies in M-theory via KSpin*, J. Geom. Phys. **58** (2008) 387, [[arXiv:0705.3484](#)] [[hep-th](#)].
- [55] H. Sati, $\mathbb{O}P^2$ bundles in M-theory, to appear in Commun. Number Theory Phys., [[arXiv:0807.4899v2](#)] [[hep-th](#)].
- [56] H. Sati, U. Schreiber and J. Stasheff, *L_∞ -connections and applications to String- and Chern-Simons n -transport*, in *Recent Developments in QFT*, eds. B. Fauser et al., Birkhäuser, Basel (2008), [[arXiv:0801.3480](#)] [[math.DG](#)].
- [57] H. Sati, U. Schreiber, and J. Stasheff, *Fivebrane structures*, to appear in Rev. Math. Phys., [[arXiv:0805.0564](#)] [[math.AT](#)].
- [58] S. Stolz, *A conjecture concerning positive Ricci curvature and the Witten genus*, Math. Ann. **304** (1996), no. 4, 785–800.
- [59] P. K. Townsend, *Four lectures on M-theory*, Summer School in High Energy Physics and Cosmology Proceedings, E. Gava et. al (eds.), Singapore, World Scientific, 1997, [[arXiv:hep-th/9612121v3](#)].
- [60] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. **B443** (1995) 85-126, [[arXiv:hep-th/9503124v2](#)].
- [61] E. Witten, *On Flux quantization in M-theory and the effective action*, J. Geom. Phys. **22** (1997) 1, [[arXiv:hep-th/9609122](#)].
- [62] E. Witten, *Five-brane effective action in M-theory*, J. Geom. Phys. **22** (1997) 103-133, [[arXiv:hep-th/9610234](#)].
- [63] E. Witten, *Duality relations among topological effects in string theory*, J. High Energy Phys. **0005** (2000) 031, [[arXiv:hep-th/9912086](#)].